ON THE EXPONENTIAL CONVERGENCE RATE FOR A NON-GRADIENT FOKKER-PLANCK EQUATION IN COMPUTATIONAL NEUROSCIENCE

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ABSTRACT. This paper concerns the proof of the exponential rate of convergence of the solution of a Fokker-Planck equation, with a drift term not being the gradient of a potential function and endowed by Robin type boundary conditions. This kind of problem arises, for example, in the study of interacting neurons populations. Previous studies have numerically shown that, after a small period of time, the solution of the evolution problem exponentially converges to the stable state of the equation.

1. INTRODUCTION

The exponential convergence of the solutions to evolution problems towards the steady states has been largely addressed for many years, see for example [1, 5, 7, 9, 10, 11, 12]. Techniques and proofs are usually based on the nature of each equation. However, the general entropy method developed in [12] for linear problems and used in computational neuroscience in [6, 8] is a powerful method for investigation of this question in the problem under consideration here.

In this work, we are concerned with the mathematical proof of the exponential rate of convergence of the solution of a non-gradient Fokker-Planck equation towards its stationary state. This problem arises, for instance, in the modeling of the evolution of the firing rates of two populations of interacting neurons. In this framework, the partial differential equation describing the evolution of the probability distribution function $p = p(t, \nu)$, with $\nu = (\nu_1, \nu_2) \in \Omega$ and $t \geq 0$ is the Fokker-Planck equation (or forward Kolmogorov equation):

$$\partial_t p + \nabla \cdot \left(Fp - \frac{\beta^2}{2} \nabla p \right) = 0, \quad \text{in } [0, \infty[\times \Omega$$

where Ω is a bounded domain of $\mathbb{R}^2$ and the vector field $F = F(\nu)$ is defined by:

$$F = \begin{pmatrix} -\nu_1 + \phi(\lambda_1 + w_{11}\nu_1 + w_{12}\nu_2) \\ -\nu_2 + \phi(\lambda_2 + w_{21}\nu_1 + w_{22}\nu_2) \end{pmatrix}$$

with $\phi(z)$ a sigmoid function and w_{ij} positive weights. The Fokker-Planck equation is endowed by the following Robin or no flux boundary conditions:

$$\left(Fp - \frac{\beta^2}{2} \nabla p \right) \cdot n = 0 \,, \text{ on } \partial\Omega$$

with n the outward normal unit vector on $\partial\Omega$, β being the noise level, and a non-negative initial data $p(0, \nu) = p_0(\nu) \geq 0$. It is easily seen that the F does not satisfy the Schwarz conditions, that is $\partial_{\nu_2} F_1 \neq \partial_{\nu_1} F_2$, so that there exists no potential function $V(\nu)$ such that $F = -\nabla V$. This in particular implies that it is less likely that there is an explicit formulation of the associated steady state, as it would be the case with a drift F defined by $F = -\nabla V$ for some smooth potential function V. We refer to [2, 3, 4] for some discussion on these issues and some cases in which, despite of the non-gradient drifts in the Fokker-Planck equations, one is able to find explicit stationary states. In our case, for general domains and so complicated drift it is in general not possible to find explicitly these stationary states. Note that as soon as we find a potential function V such that the drift $F = -\nabla V$, the steady state reads

$$p_\infty = C \exp \left(\frac{-2V}{\beta^2} \right),$$

with C a normalization coefficient.

In this work, we will deal with estimates of the exponential convergence rates for general linear Fokker-Planck equations with non-gradients drifts of the form:

(1.1)
$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (Fu) = 0 & \text{in } \Omega \times (0, T), \\ u(0, \cdot) = u_0(\cdot) & \text{in } \Omega, \\ (Fu - \nabla u) \cdot n = 0 & \text{on } \partial\Omega. \end{cases}$$

where the unknown $u(t, x)$ is a probability density function, the drift $F \in C^2(\overline{\Omega})$ satisfies the incoming boundary condition

(1.2)
$$F \cdot n < 0 \quad \text{on } \partial\Omega,$$

and the initial data is normalized to 1, $\int_\Omega u_0(x) \, dx = 1$. We have assumed unit diffusion constant for simplicity without loss of generality. Under these hypotheses, existence, uniqueness and positivity of the solution $u = u(t, x)$ of the evolution problem (1.1), and of the stable state (stationary solution) $u_\infty(x)$ of the associated problem were proved in [8, Theorem 2], as well as the mass density conservation,

$$\int_\Omega u(t, x) \, dx = \int_\Omega u_0(x) \, dx = 1,$$

and the L^2-convergence of the time dependent solution to the stationary solution u_∞. Numerical simulations were also performed underlying the exponential rate of convergence of the solution of the evolution problem towards the stable state. However, the theoretical proof of this exponential rate of convergence was not discussed in [8]. In the next section we will discuss the exponential convergence towards the steady state u_∞ by direct Poincaré's inequalities. Section 3 is devoted to an alternative approach implying relations between the sharp exponential convergence rates with the best constants for different Poincaré's type inequalities.

2. Convergence to equilibrium

Consider a drift $F \in C^2(\overline{\Omega})$ such that $F \cdot n < 0$ and let $u = u(t, x)$ be the unique solution of problem (1.1), and $u_\infty = u_\infty(x)$ be the solution of the stationary associated problem:

$$(2.1) \qquad \begin{cases} -\Delta u_\infty + \nabla(F u_\infty) = 0 & \text{in } \Omega, \\ (F u_\infty - \nabla u_\infty) \cdot n = 0 & \text{on } \partial\Omega, \end{cases}$$

ensured by [8, Theorem 2] and satisfying

$$\int_\Omega u_0(x)\, dx = \int_\Omega u_\infty(x)\, dx = 1.$$

We shall prove the following result:

Theorem 1. *The solution u to* (1.1) *exponentially converges to the steady state u_∞: there exist $\alpha, C > 0$ such that*

$$\|u - u_\infty\|_{L^2(\Omega)} \le C \exp(-\alpha t).$$

Let us first remind the reader that applying the General Entropy Method [12] adapted to this problem in [8, Theorem 4] one obtains

$$(2.2) \qquad \frac{d}{dt} \mathcal{H}(u|u_\infty) = -\mathcal{D}(u|u_\infty) \le 0,$$

where

$$\mathcal{H}(u|u_\infty) = \int_\Omega \frac{(u - u_\infty)^2}{u_\infty}\, dx$$

and

$$\mathcal{D}(u|u_\infty) = \int_\Omega u_\infty \left| \nabla \frac{u}{u_\infty} \right|^2 dx.$$

Therefore, in order to get the exponential convergence, the key point to prove is the following Poincaré inequality:

$$(2.3) \qquad \mathcal{H}(u|u_\infty) = \int_\Omega \frac{(u - u_\infty)^2}{u_\infty}\, dx \le \mathcal{P}_\Omega(u_\infty) \int_\Omega u_\infty \left| \nabla \frac{u}{u_\infty} \right|^2 dx = \mathcal{P}_\Omega(u_\infty)\, \mathcal{D}(u|u_\infty),$$

where $\mathcal{P}_\Omega(u_\infty) > 0$ is the best constant for the Poincaré inequality (2.3), with weight u_∞ in the domain Ω. In the sequel, we will not put the subindex Ω in the Poincaré constants for notational simplicity. In fact, from (2.2) and (2.3) we then deduce that

$$\frac{d}{dt} \mathcal{H}(u|u_\infty) + \mathcal{P}(u_\infty)\, \mathcal{H}(u|u_\infty) \le 0,$$

and applying Gronwall's Lemma

$$\mathcal{H}(u|u_\infty) \le C \exp(-\alpha t),$$

with $C = \mathcal{H}(u_0|u_\infty)$ and $\alpha = \mathcal{P}(u_\infty)$, leading to the exponential convergence of u to u_∞. Notice that (2.3) is equivalent to

$$(2.4) \qquad \int_\Omega u_\infty \left(\frac{u}{u_\infty} - 1 \right)^2 dx \le \mathcal{P}(u_\infty) \int_\Omega u_\infty \left| \nabla \left(\frac{u}{u_\infty} - 1 \right) \right|^2 dx,$$

Put $w = \frac{u}{u_\infty} - 1$, then (2.4) reads

$$\int_\Omega u_\infty w^2 \, dx \le \mathcal{P}(u_\infty) \int_\Omega u_\infty \, |\nabla w|^2 \, dx \, .$$

Note that, since both u and u_∞ are normalized to 1, we also have the constraint:

$$\int_\Omega u_\infty w \, dx = \int_\Omega u_\infty \left(\frac{u}{u_\infty} - 1 \right) \, dx = \int_\Omega (u - u_\infty) \, dx = 0.$$

Inequality (2.3) then follows from the following result:

Proposition 1. *Under the assumption*

$$\int_\Omega \Phi H \, dx = 0,$$

where H is bounded from above and below by positive constants, there exists a sharp constant $\mathcal{P}(H) > 0$ such that the following Poincaré inequality holds

$$\int_\Omega H |\Phi|^2 \, dx \le \mathcal{P}(H) \int_\Omega H |\nabla \Phi|^2 \, dx.$$

Proof: We follow the classical proof of Poincaré's inequality, by contradiction. Suppose that there exists a sequence $\{\Phi_m\}$, such that

$$\max(H) \int_\Omega |\Phi_m|^2 \, dx \ge \int_\Omega H |\Phi_m|^2 \, dx \ge m \int_\Omega H |\nabla \Phi_m|^2 \, dx \ge m \min(H) \int_\Omega |\nabla \Phi_m|^2 \, dx.$$

By normalization, we can suppose that $\|\Phi_m\|_{L^2} = 1$. Therefore the sequence $\{\Phi_m\}$ is bounded in $W^{1,2}(\Omega)$. By mean of the Rellich-Kondrachov Theorem, there exists a subsequence $\{\Phi_{m_j}\}$ and a function $\bar{\Phi}$ in $L^2(\Omega)$ such that $\{\Phi_{m_j}\}$ converges strongly to $\bar{\Phi}$ in $L^2(\Omega)$. Passing to the limit $m \to \infty$, we get that $\nabla \bar{\Phi} = 0$ a.e. and $\|\bar{\Phi}\|_{L^2} = 1$, which implies $\bar{\Phi}$ is a constant. But, since H is positive, this contradicts the hypothesis $\int_\Omega \bar{\Phi} H \, dx = 0$, concluding the proof. ∎

Although we can obviously relate the best constant $\mathcal{P}(u_\infty)$ to the classical Poincaré inequality best constant $\mathcal{P}(1)$ by $\mathcal{P}(u_\infty) \le \frac{\max(u_\infty)}{\min(u_\infty)} \mathcal{P}(1)$, we have no information on the value of the constant obtained by this proof. In the next section, we propose an alternative proof in which we show that the constants C and α in Theorem 1 are linked to the bounds of the solution K of an auxiliary problem.

3. Alternative Proof

In this section, we propose an alternative proof of Theorem 1 which gives us another characterization of the constants C and α. Note first that to prove the exponential convergence of u to u_∞, is equivalent to prove the exponential convergence of $\Phi = u/u_\infty$ to 1. As we will see, the proof is based on a new conservation property, see Proposition 3 below.

Before dealing with the exponential convergence problem for the function Φ, we need to define for which problem Φ is a solution. This is done in the following. Let us first define Φ and the initial data Φ_0 as follows, since $u_\infty > 0$, we define

$$\Phi(t,x) = \frac{u(t,x)}{u_\infty(x)}, \quad \Phi_0(x) = \frac{u_0(x)}{u_\infty(x)} \, ,$$

and let us introduce a modified drift F^* as

(3.1)
$$F^* = F - 2\frac{\nabla u_\infty}{u_\infty}.$$

We note that on the boundary $\partial\Omega$, we obtain

$$F^* \cdot n = \left(F - 2\frac{\nabla u_\infty}{u_\infty}\right) \cdot n = -F \cdot n > 0,$$

since $F \cdot n = (\nabla u_\infty / u_\infty) \cdot n$ and F satisfies (1.2).

Proposition 2. *The function $\Phi = u/u_\infty$ satisfies*

(3.2)
$$\begin{cases} \partial_t \Phi - \Delta \Phi + F^* \cdot \nabla \Phi = 0 \ in \ \Omega \times (0, T), \\ \Phi(., 0) = \Phi_0(.) \ in \ \Omega, \\ \nabla \Phi \cdot n = 0 \ on \ \partial\Omega. \end{cases}$$

Proof: In order to simplify the notation, consider $g = 1/u_\infty$, $g \in C^2(\Omega) \cap C(\bar{\Omega})$ and $g > 0$, and $\Phi = u\,g$ and $\Phi_0 = u_0\,g$. We derive $\Phi = u\,g$ to get, for all $i = 1, 2$

$$\partial_{x_i}\Phi = g\partial_{x_i}u + u\partial_{x_i}g,$$

$$\partial_{x_i}^2 \Phi = \partial_{x_i}^2 u\,g + \Phi\frac{\partial_{x_i}^2 g}{g} + 2\frac{\partial_{x_i}g}{g}\partial_{x_i}\Phi - 2\Phi\left|\frac{\partial_{x_i}g}{g}\right|^2.$$

These identities imply, for the boundary conditions on $\partial\Omega$

$$\nabla\Phi \cdot n = \left(\frac{\nabla u}{u_\infty} - \frac{u\nabla u_\infty}{u_\infty^2}\right) \cdot n = \left(\frac{\nabla u}{u_\infty} - \frac{uF}{u_\infty}\right) \cdot n = 0,$$

and inside the domain Ω, we obtain

$$\partial_t \Phi - \Delta\Phi = (\partial_t u - \Delta u)g - \Phi\frac{\Delta g}{g} - 2\nabla\Phi \cdot \frac{\nabla g}{g} + 2\Phi\left|\frac{\nabla g}{g}\right|^2$$

$$= -\nabla \cdot (Fu)g - \Phi\frac{\Delta g}{g} - 2\nabla\Phi \cdot \frac{\nabla g}{g} + 2\Phi\left|\frac{\nabla g}{g}\right|^2$$

$$= -\nabla \cdot (F\Phi) - 2\nabla\Phi \cdot \frac{\nabla g}{g}$$

(3.3)
$$+ \Phi\left[-\left(\frac{\Delta g}{g} - 2\left|\frac{\nabla g}{g}\right|^2\right) + F \cdot \frac{\nabla g}{g}\right].$$

Since $u_\infty = 1/g$, we have

$$\nabla u_\infty = -\frac{\nabla g}{g^2} \quad \text{and} \quad \Delta u_\infty = -\frac{\Delta g}{g^2} + 2\frac{|\nabla g|^2}{g^3},$$

so that, dividing (3.3) by $g = 1/u_\infty$, we get

$$u_\infty\partial_t\Phi - u_\infty\Delta\Phi + u_\infty F \cdot \nabla\Phi - 2\nabla\Phi \cdot \nabla u_\infty$$
$$+ \Phi[-\Delta u_\infty + F \cdot \nabla u_\infty + u_\infty\nabla \cdot F] = 0.$$

Hence Φ must satisfy

(3.4)
$$\begin{cases} u_\infty \partial_t \Phi - u_\infty \Delta \Phi + u_\infty F \cdot \nabla \Phi - 2\nabla \Phi \cdot \nabla u_\infty \\ \qquad\qquad + \Phi[-\Delta u_\infty + \nabla \cdot (Fu_\infty)] = 0 \quad \text{in } \Omega \times (0,T), \\ \Phi(\cdot,0) = \Phi_0(\cdot) \quad \text{in } \Omega, \\ \nabla \Phi \cdot n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Finally, recalling the definition of F^* given by (3.1), and that u_∞ solves (2.1), problem (3.4) is converted into (3.2), concluding the proof. ∎

Consider now the stationary problem associated to (3.2):

(3.5)
$$\begin{cases} -\Delta \Phi_\infty + F^* \cdot \nabla \Phi_\infty = 0 \quad \text{in } \Omega \times (0,T), \\ \nabla \Phi_\infty \cdot n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

then it is easily seen that the constants are solutions to (3.5), so that proving the exponential convergence of Φ to $\Phi_\infty = 1$, will give us the wanted exponential convergence of u to u_∞. So, the main result we have to prove, yielding to the exponential convergence of u to u_∞ is the following:

Theorem 2. *The solution Φ of* (3.2) *exponentially converges in time to its equilibrium state Φ_∞, solution to* (3.5).

In the sequel, in order to prove Theorem 2, we need an auxiliary problem, which is the dual problem to (3.5) given by

(3.6)
$$\begin{cases} \Delta K + \nabla \cdot (KF^*) = 0 \quad \text{in } \Omega, \\ (\nabla K + KF^*) \cdot n = 0 \quad \text{on } \partial\Omega, \end{cases}$$

for which we next prove a conservation property of the function ΦK. This conservation results is a key point in order to prove a Poincaré's inequality on the function Φ and to get in an alternative way the exponential convergence.

Proposition 3. *Let K be a solution of* (3.6) *and Φ be a solution of* (3.2)*, then*

$$\int_\Omega \Phi_0 K \, dx = \int_\Omega \Phi K \, dx.$$

Proof: Note that $-F^* \cdot n < 0$ on $\partial\Omega$, thus by using the same arguments as in [8], this problem admits a unique positive and bounded solution $K \in H^2(\Omega)$. Furthermore, we have

$$-\Delta \Phi + F^* \nabla \Phi = -\frac{1}{K} \nabla \cdot (K \nabla \Phi) + \left(\frac{\nabla K}{K} + F^* \right) \nabla \Phi,$$

so that problem (3.2) can be written as

(3.7)
$$\begin{cases} \partial_t \Phi - \frac{1}{K} \nabla \cdot (K \nabla \Phi) + \left(\frac{\nabla K}{K} + F^* \right) \cdot \nabla \Phi = 0 \quad \text{in } \Omega \times (0,T), \\ \Phi(.,0) = \Phi_0(.) \quad \text{in } \Omega, \\ \nabla \Phi \cdot n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Use K as a test function for (3.7), then

$$0 = \partial_t \int_\Omega \Phi K \, dx + \int_\Omega (\nabla K + K F^*) \cdot \nabla \Phi \, dx - \int_{\partial\Omega} K \nabla \Phi \cdot n \, d\sigma(x)$$

$$= \partial_t \int_\Omega \Phi K \, dx - \int_\Omega \nabla \cdot (\nabla K + K F^*) \Phi \, dx + \int_{\partial\Omega} \Phi (\nabla K + K F^*) \cdot n \, d\sigma(x)$$

$$= \partial_t \int_\Omega \Phi K \, dx - \int_\Omega (\Delta K + \nabla \cdot (K F^*)) \Phi \, dx = \partial_t \int_\Omega \Phi K \, dx,$$

which concludes the proof. ∎

Since, problem (3.6) admits a unique solution, then the constants are the unique solutions, in distributional sense, of the stationary problem (3.5). Hence, up to a normalization constant, we shall consider in the sequel that $\Phi_\infty = 1$. Moreover, if we consider $\Psi = \Phi - \Phi_\infty$, then Ψ still satisfies (3.2) and converges to 0, in other words its equilibrium is the null function. Therefore, renaming $\Psi = \Phi$ and in order to simplify the proof of Theorem 2, we are reduced to prove the exponential convergence of Φ solution of (3.2) to 0, which is done in Theorem 3 below. We impose the following normalization

$$\int_\Omega \Phi_0 K \, dx = 0,$$

so that from the conservation property in Proposition 3 we get

$$\int_\Omega \Phi K \, dx = 0.$$

Thanks to this normalization, the proof of Theorem 2 is reduced to prove the following result.

Theorem 3. *Assuming that*

$$\int_\Omega \Phi_0 K \, dx = 0,$$

then, the solution Φ of (3.2) exponentially decays in time towards 0, that is,

$$\|\Phi\|_{L^2(\Omega)} \leq \tilde{C} \exp(-\tilde{\alpha} t),$$

where $\tilde{C}$ and $\tilde{\alpha}$ are estimated by

$$\tilde{C} = \frac{1}{\min(K)} \int_\Omega |\Phi_0|^2 K \, dx \quad and \quad \tilde{\alpha} = \mathcal{P}(K).$$

Proof: Use $K\Phi$ as a test function for (3.7). Since K is bounded from above and from below by some positive constants, and since from the normalization hypothesis, the Poincaré

inequality of Proposition 1 holds, then:

$$
\begin{aligned}
0 &= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\, dx + \int_\Omega |\nabla\Phi|^2 K\, dx + \int_\Omega \Phi(\nabla K + KF^*)\cdot\nabla\Phi\, dx \\
&= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\, dx + \int_\Omega |\nabla\Phi|^2 K\, dx + \frac{1}{2}\int_\Omega (\nabla K + KF^*)\cdot\nabla|\Phi|^2\, dx \\
&= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\, dx + \int_\Omega |\nabla\Phi|^2 K\, dx - \frac{1}{2}\int_\Omega (\Delta K + \nabla\cdot(KF^*))|\Phi|^2\, dx \\
&= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\, dx + \int_\Omega |\nabla\Phi|^2 K\, dx
\end{aligned}
$$

$$
(3.8) \qquad \geq \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\, dx + \mathcal{P}(K)\int_\Omega |\Phi|^2 K\, dx\,.
$$

Inequality (3.8) implies:

$$
\int_\Omega |\Phi|^2 K\, dx \leq \left(\int_\Omega |\Phi_0|^2 K\, dx\right)\exp(-2\tilde{\alpha}t),
$$

with $\tilde{\alpha} = \mathcal{P}(K)$. With some more computations, we finally obtain

$$
\int_\Omega |\Phi|^2\, dx \leq \frac{1}{\min(K)}\int_\Omega |\Phi|^2 K\, dx \leq \left(\frac{1}{\min(K)}\int_\Omega |\Phi_0|^2 K\, dx\right)\exp(-2\tilde{\alpha}t),
$$

concluding the proof of theorem 3. ∎

Finally, to conclude the proof of the time exponential convergence of u, solution to (1.1), to the steady state u_∞ given by (2.1), we just have to note that, since $g = 1/u_\infty$ is strictly positive and bounded, Theorem 2 leads to:

$$
\|u - u_\infty\|_{L^2(\Omega)} \leq \max(u_\infty)\tilde{C}\exp(-\tilde{\alpha}t)\,,
$$

concluding the proof of Theorem 1. Notice that the decay constants might be different to the ones in Theorem 1 obtained in Section 2 due to the different stationary problems with solutions K and u_∞ used.

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