JEPE Vol 1, 2015, p. 231-418

Journal of Elliptic and Parabolic Equations

Orthogonal $\perp$ *Editions*

Journal of Elliptic and Parabolic Equations

JEPE Vol 1, 2015, p. 231-241

Orthogonal $\perp$

EXISTENCE OF STANDING WAVES SOLUTION FOR A NONLINEAR SCHRÖDINGER EQUATION IN $\mathbb{R}^N$

CLAUDIANOR O. ALVES

ABSTRACT. In this paper, we investigate the existence of a positive solution for the following class of elliptic equation

$$-\epsilon^2 \Delta u + V(x)u = f(u) \ \text{ in } \mathbb{R}^N,$$

where $\epsilon > 0$ is a positive parameter, f has a subcritical growth and V is a positive potential verifying some conditions.

1. INTRODUCTION

In recent years, many authors have considered the existence of solution for the following class of elliptic equation

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u = f(u) & \text{in } \mathbb{R}^N, \\ u > 0 \ \text{ in } \ \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$(P)_\epsilon$}$$

where $\epsilon > 0$ is a positive parameter, $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions with V being a nonnegative function and f having a subcritical or critical growth. The existence and concentration of positive solutions for general semilinear elliptic equations $(P)_\epsilon$ for the case $N \geq 2$ have been extensively studied, see for example, Ackermann and Szulkin [2], Alves, do Ó and Souto [4], Bartsch, Pankov and Wang [5], do Ó and Souto [9], del Pino and Felmer [6, 8], del Pino, Felmer and Miyagaki [8], Floer and Weinstein [10], Oh [11], Rabinowitz [12], Wang [13] and their references.

The knowledge of the solutions of $(P)_\epsilon$ has a great importance for studying the existence of *standing wave solutions* for, the nonlinear Schrödinger equation

$$i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + W(z)\Psi - f(\Psi) \ \text{ for all } \ z \in \mathbb{R}^N, \tag{NLS}$$

which are solutions of the form $\Psi(x,t) = \exp(-iEt/\epsilon)u(x)$, where u is a solution of $(P)_\epsilon$. The equation (NLS) is one of the main objects of the quantum physics, because it appears in problems involving nonlinear optics, plasma physics and condensed matter physics, see [10] and [11] for more details about these topics.

2010 *Mathematics Subject Classification.* Primary: 35J20,; Secondary: 35J65.
Key words and phrases. superlinear problem, positive solution, variational methods.
Received 17/08/2015, accepted 12/11/2015.
Research of C. O. Alves partially supported by CNPq 304036/2013-7 and INCT-MAT.

In a seminal paper, Rabinowitz [12] introduced the following condition on V

$$0 < \inf_{z \in \mathbb{R}^N} V(z) < \liminf_{|z| \to +\infty} V(z). \tag{V_8}$$

Later Wang [13] showed that these solutions concentrate at global minimum points of V as ϵ tends to 0.

In [6], del Pino and Felmer established the existence of positive solutions which concentrate around local minimum of V, by introducing a penalization method. More precisely, they assumed that there is an open and bounded set $\mathcal{O}$ compactly contained in $\mathbb{R}^N$ such that

$$0 < \gamma \leq V_0 = \inf_{z \in \mathcal{O}} V(z) < \min_{z \in \partial\mathcal{O}} V(z). \tag{V_1}$$

Motivated by this result, Alves, do Ó and Souto [4] and do Ó and Souto [9] studied the same type of problem with f having critical growth for $N \geq 3$ and exponential critical growth for $N = 2$ respectively.

In [7], del Pino, Felmer and Miyagaki considered the case where potential V has a geometry like saddle, essentially they assumed the following conditions on V: First of all, they fixed two subspaces $X, Y \subset \mathbb{R}^N$ such that

$$\mathbb{R}^N = X \oplus Y.$$

By supposing that V is bounded, they fixed $c_0, c_1 > 0$ satisfying

$$c_0 = \inf_{z \in \mathbb{R}^N} V(z) > 0$$

and

$$c_1 = \sup_{x \in X} V(x).$$

Furthermore, they also supposed that $V \in C^2(\mathbb{R}^N)$ and it verifies the following geometric conditions:

(V_1)

$$c_0 = \inf_{R>0} \sup_{x \in \partial B_R(0) \cap X} V(x) < \inf_{y \in Y} V(y).$$

(V_2) The functions V, $\frac{\partial V}{\partial x_i}$ and $\frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^N$ for all $i, j \in \{1, ..., N\}$.

(V_3) V satisfies the Palais-Smale condition, that is, if $(x_n) \subset \mathbb{R}^N$ is a sequence such that $(V(x_n))$ is bounded and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence in $\mathbb{R}^N$.

Using the above conditions on V, and supposing that

$$c_1 < 2^{\frac{2(p-1)}{N+2-p(N-2)}} c_0,$$

del Pino, Felmer and Miyagaki showed the existence of positive solutions for the following problem

$$-\epsilon^2 \Delta u + V(z)u = |u|^{p-2}u \ \text{ in } \ \mathbb{R}^N,$$

where $p \in (2, 2^*)$ if $N \geq 3$ and $p \in (2, +\infty)$ if $N = 1, 2$, for $\epsilon > 0$ small enough. The main tool used was the variational method, more precisely, the authors found critical points of the functional

$$E_\epsilon(u) = \int_{\mathbb{R}^N} (\epsilon^2|\nabla u|^2 + |u|^2)\, dx$$

on the manifold

$$\mathcal{M} = \left\{ u \in H^1(\mathbb{R}^N) \cap P \, : \, \int_{\mathbb{R}^N} |u|^p \, dx = 1 \right\},$$

where P denotes the cone of nonnegative functions of $H^1(\mathbb{R}^N)$. Recently, in [3], Alves has studied the same type of problem with f having an exponential critical growth and $N = 2$.

In [8], del Pino and Felmer have considered the following assumptions on V:

(V_4) V is of class C^1 and there exists an $\alpha > 0$ such that

$$V(z) \geq \alpha, \quad \forall z \in \mathbb{R}^N.$$

Locally, it was fixed an open and bounded set $D \subset \mathbb{R}^N$ and subsets $B_0, B \subset D$ with B connected. Using these sets, we denoted by Γ the class of all continuous functions $\phi : B \to D$ with the property that $\phi(y) = y$ for all $y \in B_0$. Define the min-max value c as

$$c = \inf_{\phi \in \Gamma} \sup_{y \in B} V(\phi(y)), \tag{1.1}$$

and assume additionally

(V_5)

$$\sup_{y \in B_0} V(y) < c.$$

(V_6) For all $\phi \in \Gamma$, $\phi(B) \cap \{y \in D \, : \, V(y) \geq c\} \neq \emptyset$.

(V_7) For all $y \in \partial D$ such that $V(y) = c$, one has $\partial_\nu V(y) \neq 0$, where ∂_ν denotes the tangential derivative.

Motivated by the above papers, in the present article we show the existence of solution for $(P)_\epsilon$, by considering two new class of potential V, namely :

Class 1: The potential V verifies the Palais-Smale condition.

(A_0) There exists a $V_0 > 0$ such that $V(x) \geq V_0, \quad \forall x \in \mathbb{R}^N$.

(A_1) $V \in C^2(\mathbb{R}^N)$ and V, $\frac{\partial V}{\partial x_i}$ and $\frac{\partial^2 V}{\partial x_i \partial x_j}$ are bounded in $\mathbb{R}^N$ for all $i, j \in \{1, ..., N\}$.

(A_2) V verifies the Palais-Smale condition, that is, if $(x_n) \subset \mathbb{R}^N$ is a sequence such that $(V(x_n))$ is bounded and $\nabla V(x_n) \to 0$, then (x_n) possesses a convergent subsequence in $\mathbb{R}^N$.

Class 2: The potential V does not have critical point on the boundary of some bounded domain.

In this class of potential, we suppose that V verifies $(A_0) - (A_1)$ and the following additional condition:

(A_3) There is a bounded domain $\Lambda \subset \mathbb{R}^N$, such that $\nabla V(x) \neq 0$ for all $x \in \partial\Lambda$.

Related to the function f, we assume that

(f_1) $\displaystyle\limsup_{s \to 0^+} \frac{f(s)}{s} = 0.$

(f_2) There exists $p \in (2, 2^*)$, such that

$$\limsup_{s \to +\infty} \frac{f(s)}{s^{p-1}} = 0;$$

(f_3) There exists $\theta > 2$ such that

$$0 < \theta F(s) \leq s f(s) \qquad \forall s > 0;$$

where $F(s) = \int_0^s f(t)\, dt$.

The statement of our main result is the following

Theorem 1. *Suppose that V belongs to the Class 1 or 2 and f satisfies $(f_1) - (f_3)$. Then, the problem $(P)_\epsilon$ has a positive solution for $\epsilon > 0$ small enough.*

In the proof of the Theorem 1, we will use variational methods, more precisely, the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1] combined with some arguments developed by del Pino and Felmer [6], for more details see Section 2.

The paper is organized as follows. In the next section, inspired by [6], we study the existence of a solution for a class of auxiliary problems. In Section 3, we prove the main theorem supposing that V belongs to Class 1, while in Section 4, we consider the case which V belongs to Class 2. Finally, in Section 5, we make some final considerations for elliptic problems with f having critical growth for $N \geq 3$ and exponential critical growth for $N = 2$.

2. del Pino and Felmer's approach

In this section, following an idea found in del Pino and Felmer [6], we will study the existence of a solution for a special class of elliptic problems associated with $(P)_\epsilon$.

Since we intend to prove the existence of positive solutions, hereafter we consider

$$f(s) = 0, \quad \forall s \leq 0.$$

Using the change of variable $v(x) = u(\epsilon x)$, it is possible to prove that $(P)_\epsilon$ is equivalent to the following problem

$$\begin{cases} -\Delta u + V(\epsilon x) u = f(u) & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{$(P)'_\epsilon$}$$

Therefore, in what follows, we prove the existence of a positive solution for $(P)'_\epsilon$. For this purpose, we start observing that from (A_0), we can work in $H^1(\mathbb{R}^N)$ with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2) dx \right)^{\frac{1}{2}},$$

which is equivalent to the usual norm.

The Euler-Lagrange functional associated with $(P)'_\epsilon$ is given by

$$I_\epsilon(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(u) dx, \ \ \forall u \in H^1(\mathbb{R}^N).$$

From the conditions on f, the functional $I_\epsilon \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and its Gâteaux derivative is

$$I'_\epsilon(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(\epsilon x) uv) dx - \int_{\mathbb{R}^N} f(u) v dx, \ \ \forall u, v \in H^1(\mathbb{R}^N).$$

It is easy to check that the critical points of I_ϵ are weak solutions of $(P)'_\epsilon$.

In the sequel, let us denote by $I_\infty : H^1(\mathbb{R}^N) \to \mathbb{R}$ the functional

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty |u|^2)dx - \int_{\mathbb{R}^N} F(u)dx,$$

where

$$V_\infty = \max_{x\in\mathbb{R}^N} V(x).$$

Furthermore, let us denote by c_∞ the mountain pass level associated with I_∞, that is,

$$c_\infty = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I_\infty(\gamma(1)) < 0\}. \tag{2.1}$$

Here, we would like to point out that c_∞ depends only on V_∞, θ and f.

2.1. **An auxiliary problem.** Given a bounded domain $\Omega \subset \mathbb{R}^N$, we fix the numbers $k = \frac{2\theta}{\theta-2} > 2$ and $a > 0$ be the value at which

$$\frac{f(a)}{a} = \frac{V_0}{k},$$

where $V_0 > 0$ is given in (A_0). Using these numbers, we set the functions

$$\tilde{f}(s) = \begin{cases} 0, & s \le 0, \\ f(s), & 0 \le s \le a, \\ \frac{V_0}{k}s, & s \ge a \end{cases}$$

and

$$g(x,s) = \chi(x)f(s) + (1-\chi)\tilde{f}(s), \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

where χ denotes the characteristic function associated with Ω, that is,

$$\chi(x) = \begin{cases} 1, & x \in \Omega \\ 0, & x \in \Omega^c. \end{cases}$$

Using the above functions, we will study the existence of a positive solution for the following problem

$$(AP)_\epsilon \qquad \begin{cases} -\Delta u + V(\epsilon x)u = g_\epsilon(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where

$$g_\epsilon(x,s) = g(\epsilon x, s), \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

The above problem is strongly related to $(P)'_\epsilon$, because if u is a solution of $(AP)_\epsilon$ verifying

$$u(x) < a, \quad \forall x \in \mathbb{R}^N \setminus \Omega_\epsilon,$$

where $\Omega_\epsilon = \Omega/\epsilon$, then u will be a solution for $(P)'_\epsilon$.

Associated with $(AP)_\epsilon$, we have the energy functional $J_\epsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J_\epsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2)\, dx - \int_{\mathbb{R}^N} G_\epsilon(x,u)\, dx,$$

where

$$G_\epsilon(x,s)=\int_0^s g_\epsilon(x,t)\,dt, \quad \forall (x,s)\in\mathbb{R}^N\times\mathbb{R}.$$

From the conditions on f, and hence on g, $J_\epsilon\in C^1(H^1(\mathbb{R}^N),\mathbb{R})$ and one has

$$J'_\epsilon(u)v=\int_{\mathbb{R}^N}(\nabla u\nabla v+V(\epsilon x)uv)dx-\int_{\mathbb{R}^N} g_\epsilon(x,u)vdx,\ \forall u,v\in H^1(\mathbb{R}^N).$$

Thus, critical points of J_ϵ correspond to weak solutions of $(AP)_\epsilon$.

Repeating the same arguments found in [6], it is easy to see that J_ϵ verifies the hypotheses of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [1] for all $\epsilon>0$. Therefore, there exists a $u_\epsilon\in H^1(\mathbb{R}^N)$ such that

$$J_\epsilon(u_\epsilon)=c_\epsilon>0 \quad \text{and} \quad J'_\epsilon(u_\epsilon)=0,$$

where

$$c_\epsilon=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J_\epsilon(\gamma(t))$$

with

$$\Gamma=\{\gamma\in C([0,1],H^1(\mathbb{R}^N)):\gamma(0)=0 \ \text{and}\ J_\epsilon(\gamma(1))<0\}.$$

Observing that

$$J_\epsilon(u)\le I_\infty(u) \quad \forall u\in H^1(\mathbb{R}^N),$$

we have

$$c_\epsilon\le c_\infty, \quad \forall\epsilon>0. \tag{2.2}$$

The lemma below establishes an important estimate from above for the H^1-norm of the family (u_ϵ).

Lemma 1. *For all $\epsilon>0$, the solution u_ϵ of $(AP)_\epsilon$ satisfies the estimate*

$$\|u_\epsilon\|^2\le 2kc_\infty.$$

Proof: Using the fact that u_ϵ is a critical point of J_ϵ, we must have

$$c_\epsilon=J_\epsilon(u_\epsilon)=J_\epsilon(u_\epsilon)-\frac{1}{\theta}J'_\epsilon(u_\epsilon)u_\epsilon\ge\frac{(\theta-2)}{4\theta}\|u_\epsilon\|^2\ge\frac{1}{2k}\|u_\epsilon\|^2.$$

Now, the result follows by combining the above inequality with (2.2). □

Here, we would like to point out that in Lemma 1, the norm $\|u_\epsilon\|$ is bounded from above, by a constant that depends only on V_∞, θ and f, then the constant does not depend on $\epsilon>0$.

3. Proof of Theorem 1: The Class 1

In this section, we will prove the Theorem 1, by supposing that V belongs to Class 1. For this purpose we will use the results obtained in Section 2 fixing

$$\Omega=B_{R_\epsilon}(0),$$

where $R_\epsilon=\frac{1}{\epsilon}$. Consequently, we know that there is a solution $u_\epsilon\in H^1(\mathbb{R}^N)$ for $(AP)_\epsilon$.

In what follows, our goal is to prove that there is $\epsilon_0>0$ such that

$$u_\epsilon(x)<a, \quad \forall x\in\mathbb{R}^N\setminus B_{\frac{R_\epsilon}{\epsilon}}(0) \quad \text{and} \quad \forall\epsilon\in(0,\epsilon_0).$$

Lemma 2. *The function u_ϵ verifies the following estimate*

$$\max_{x\in\partial B_{\frac{R_\epsilon}{\epsilon}}(0)} u_\epsilon(x) \to 0, \quad \epsilon \to 0.$$

Proof: Assume by contradiction that there exists an $\epsilon_n \to 0$ and $\gamma > 0$ such that

$$\max_{x\in\partial B_{\frac{R_{\epsilon_n}}{\epsilon_n}}(0)} u_n(x) \geq \gamma \quad \forall n \in \mathbb{N},$$

where $u_n = u_{\epsilon_n}$. From now on, we fix $x_n \in \partial B_{\frac{R_{\epsilon_n}}{\epsilon_n}}(0)$ satisfying

$$u_n(x_n) = \max_{x\in\partial B_{\frac{R_{\epsilon_n}}{\epsilon_n}}(0)} u_{\epsilon_n}(x).$$

Therefore,

$$u_n(x_n) \geq \gamma, \quad \forall n \in \mathbb{N}.$$

By Lemma 1, (u_n) is bounded in $H^1(\mathbb{R}^N)$. Thereby, setting $w_n = u_n(\cdot + x_n)$, we can guarantee that (w_n) is also bounded in $H^1(\mathbb{R}^N)$ and it satisfies

$$\begin{cases} -\Delta w_n + V(\epsilon_n x + \epsilon_n x_n) w_n = g(\epsilon_n x + \epsilon_n x_n, w_n), x \in \mathbb{R}^N, \\ \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Using bootstrap arguments, it is possible to show that (w_n) converges uniformly on compact set to its weak limit $w \in H^1(\mathbb{R}^N)$. Then, $w \in C(\mathbb{R}^N)$ and $w(0) \geq \gamma$, which implies that $w \neq 0$. Moreover, by (A_1), there exists a subsequence of $(\epsilon_n x_n)$, still denoted by $(\epsilon_n x_n)$, such that

$$\alpha_1 = \lim_{n\to+\infty} V(\epsilon_n x_n),$$

for some $\alpha_1 > 0$. Since for each $\phi \in H^1(\mathbb{R}^N)$, the equality below holds

$$\int_{\mathbb{R}^N} \nabla w_n \nabla\phi \, dx + \int_{\mathbb{R}^N} V(\epsilon_n x + \epsilon_n x_n) w_n \phi \, dx - \int_{\mathbb{R}^N} g(\epsilon_n x + \epsilon_n x_n, w_n)\phi \, dx = o_n(1)\|\phi\|,$$

taking the limit of $n \to +\infty$, let us deduce that w is a nontrivial solution of the problem

$$\Delta u - \alpha_1 u + \tilde{g}(x, u) = 0, \quad x \in \mathbb{R}^N, \tag{3.1}$$

where

$$\tilde{g}(x, s) = \tilde{\chi}(x) f(s) + (1 - \tilde{\chi}(x))\tilde{f}(s),$$

for some $\tilde{\chi} \in L^\infty(\mathbb{R}^N)$. Thus, by regularity theory, $w \in L^\infty(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

For each $j \in \mathbb{N}$, there is $\phi_j \in C_0^\infty(\mathbb{R}^N)$ such that

$$\|\phi_j - w\| \leq 1/j,$$

that is,

$$\|\phi_j - w\| = o_j(1).$$

Using $\frac{\partial\phi_j}{\partial x_i}$ as a test function, we get

$$\int_{\mathbb{R}^N} \nabla w_n \nabla \frac{\partial\phi_j}{\partial x_i} \, dx + \int_{\mathbb{R}^N} V(\epsilon_n x + \epsilon_n x_n) w_n \frac{\partial\phi_j}{\partial x_i} \, dx - \int_{\mathbb{R}^N} g(\epsilon_n x + \epsilon_n x_n, w_n) \frac{\partial\phi_j}{\partial x_i} \, dx = o_n(1).$$

Now, applying the Lebesgue's Theorem, we deduce that

$$\int_{\mathbb{R}^N} \nabla w_n \nabla \frac{\partial\phi_j}{\partial x_i} \, dx = \int_{\mathbb{R}^N} \nabla w \nabla \frac{\partial\phi_j}{\partial x_i} \, dx + o_n(1)$$

and

$$\int_{\mathbb{R}^N} g(\epsilon_n x + \epsilon_n x_n, w_n) \frac{\partial \phi_j}{\partial x_i}\, dx = \int_{\mathbb{R}^N} \tilde{g}(x, w) \frac{\partial \phi_j}{\partial x_i}\, dx + o_n(1).$$

From the above limit with (3.1), we find

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) w_n \frac{\partial \phi_j}{\partial x_i}\, dx \right| = 0.$$

As ϕ_j has compact support, the above limit gives

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) w \frac{\partial \phi_j}{\partial x_i}\, dx \right| = 0.$$

Now, recalling that $\frac{\partial w}{\partial x_i} \in L^2(\mathbb{R}^N)$, we have that $(\frac{\partial \phi_j}{\partial x_i})$ is bounded in $L^2(\mathbb{R}^N)$. Hence,

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) \phi_j \frac{\partial \phi_j}{\partial x_i}\, dx \right| = o_j(1),$$

and so,

$$\limsup_{n \to +\infty} \left| \frac{1}{2} \int_{\mathbb{R}^N} (V(\epsilon_n x + \epsilon_n x_n) - V(\epsilon_n x_n)) \frac{\partial (\phi_j^2)}{\partial x_i}\, dx \right| = o_j(1).$$

Using Green's Theorem together with the fact that ϕ_j has compact support, we obtain the limit below

$$\limsup_{n \to +\infty} \left| \int_{\mathbb{R}^N} \frac{\partial V}{\partial x_i}(\epsilon_n x + \epsilon_n x_n)\, \phi_j^2 \right| dx = o_j(1),$$

which combined with (A_1) leads to

$$\limsup_{n \to +\infty} \left| \frac{\partial V}{\partial x_i}(\epsilon_n x_n) \int_{\mathbb{R}^N} |\phi_j|^2\, dx \right| = o_j(1).$$

As

$$\int_{\mathbb{R}^2} |\phi_j|^2\, dx \to \int_{\mathbb{R}^N} |w|^2\, dx > 0 \quad \text{as} \quad j \to +\infty,$$

it follows that

$$\limsup_{n \to +\infty} \left| \frac{\partial V}{\partial x_i}(\epsilon_n x_n) \right| = o_j(1), \quad \forall i \in \{1, ..., N\}.$$

Since j is arbitrary, we derive that

$$\nabla V(\epsilon_n x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, $(\epsilon_n x_n)$ is a $(PS)_{\alpha_1}$ sequence for V, which is absurd, since by (A_2), V satisfies the (PS) condition and $(\epsilon_n x_n)$ does not have any convergent subsequence in $\mathbb{R}^N$, because

$$|\epsilon_n x_{\epsilon_n}| = R_{\epsilon_n} = \frac{1}{\epsilon_n} \to +\infty, \quad \text{as} \quad n \to +\infty.$$

□

Proof of Theorem 1 (conclusion): From Lemma 2, there exists an $\epsilon_0 > 0$ such that

$$\max_{x \in \partial B_{\frac{R_\epsilon}{\epsilon}}(0)} u_\epsilon(x) < a, \quad \forall \epsilon \in (0, \epsilon_0).$$

Considering the function

$$\tilde{u}_\epsilon(x) = \begin{cases} 0, & x \in \overline{B}_{\frac{R_\epsilon}{\epsilon}}(0), \\ (u_\epsilon - a)^+(x), & x \in \mathbb{R}^N \setminus B_{\frac{R_\epsilon}{\epsilon}}(0), \end{cases}$$

it follows that $\tilde{u}_\epsilon \in H^1(\mathbb{R}^N)$. Thereby, $J'_\epsilon(u_\epsilon)\tilde{u}_\epsilon = 0$, or equivalently,

$$\int_{\mathbb{R}^N} \nabla u_\epsilon \nabla \tilde{u}_\epsilon \, dx + \int_{\mathbb{R}^N} V(\epsilon x) u_\epsilon \tilde{u}_\epsilon \, dx = \int_{\mathbb{R}^N} g_\epsilon(x, u_\epsilon)\tilde{u}_\epsilon \, dx.$$

Now, using the definition of g_ϵ, it is possible to prove that $\tilde{u}_\epsilon \equiv 0$. From this,

$$u_\epsilon(x) \le a, \quad \forall x \in \mathbb{R}^N \setminus B_{\frac{R_\epsilon}{\epsilon}}(0),$$

showing that u_ϵ is a solution for $(P)'_\epsilon$. □

4. Proof of Theorem 1: The Class 2

In this section, we will prove the Theorem 1 for the case that V belongs to Class 2. However, we will use the results showed in Section 2 with

$$\Omega = \Lambda.$$

Then, we also have a solution $u_\epsilon \in H^1(\mathbb{R}^N)$ for $(AP)_\epsilon$.

Next, we will show that there exists an $\epsilon_0 > 0$ such that

$$u_\epsilon(x) < a, \quad \forall x \in \mathbb{R}^N \setminus \Lambda_\epsilon \quad \text{and} \quad \forall \epsilon \in (0, \epsilon_0).$$

Lemma 3. *The function u_ϵ verifies the following convergence*

$$\max_{x \in \partial\Lambda_\epsilon} u_\epsilon(x) \to 0, \quad \epsilon \to 0.$$

Proof: Using the same type of arguments explored in the proof of Lemma 2, we find a sequence $(x_n) \subset \partial\Lambda_{\epsilon_n}$, with $\epsilon_n \to 0$, satisfying

$$\nabla V(\epsilon_n x_n) \to 0.$$

Since $(\epsilon_n x_n) \subset \partial\Lambda$, and $\partial\Lambda$ is a compact set in $\mathbb{R}^N$, we can assume that there is $x_0 \in \partial\Lambda$ such that

$$\epsilon_n x_n \to x_0 \quad \text{in} \quad \mathbb{R}^N.$$

Gathering the above limits with the fact that $V \in C^1(\mathbb{R}^N, \mathbb{R})$, we get

$$x_0 \in \partial\Lambda \quad \text{and} \quad \nabla V(x_0) = 0,$$

contradicting (A_3). □

Proof of Theorem 1 (conclusion): The conclusion of the proof follows as in Section 3.

5. Final considerations

In this section, we would like to point out that the arguments explored in the present paper can be applied to study the existence of solution for elliptic problems with critical growth for $N \geq 3$ and exponential critical growth for $N = 2$, with the following hypotheses:

Critical growth for $N \geq 3$:

$$\left\{ \begin{array}{l} -\epsilon^2 \Delta u + V(x)u = \lambda |u|^{q-2}u + |u|^{2^*-2}u \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{array} \right. \qquad (P_\epsilon)_*$$

where $\epsilon, \lambda > 0$ are positive parameters, $q \in (2, 2^*)$ and $2^* = \frac{2N}{N-2}$.

The main result associated with this class of problem is the following

Theorem 2. *Assume that V belongs to Class 1 or 2. Then, there is $\epsilon_0 > 0$ such that*
a) If $N \geq 4$, $(P_\epsilon)_*$ has a positive solution for all $\epsilon \in (0, \epsilon_0]$ and $\lambda > 0$.
b) If $N = 3$, there exists an $\lambda^* > 0$, which is independent of $\epsilon_0 > 0$, such that $(P_\epsilon)_*$ has a positive solution for all $\epsilon \in (0, \epsilon_0]$ and $\lambda \geq \lambda^*$.

In the proof of this result, we adapt the arguments found in [4], because in that paper, the problem $(P_\epsilon)_*$ has been considered with the same condition on V, as considered in [6].

Critical growth for $N = 2$:

Hereafter, $f \in C^1(\mathbb{R})$ and it satisfies the following conditions:

(f_1) There is $C > 0$ such that
$$|f(s)| \leq Ce^{4\pi |s|^2} \quad \text{for all} \quad s \in \mathbb{R}.$$

(f_2) $\displaystyle\lim_{s \to 0} \frac{f(s)}{s} = 0.$

(f_3) There is $\theta > 2$ such that
$$0 < \theta F(s) := \theta \int_0^s f(t)dt \leq sf(s), \quad \text{for all} \quad s \in \mathbb{R} \setminus \{0\}.$$

(f_4) There exist constants $p > 2$ and $C_p > 0$ such that
$$f(s) \geq C_p s^{p-1} \quad \text{for all} \quad s > 0,$$
where
$$C_p > \left[\beta_p \left(\frac{2\theta}{\theta - 2} \right) \frac{1}{\min\{1, V_0\}} \right]^{(p-2)/2},$$
with
$$\beta_p = \inf_{\mathcal{N}_\infty} J_\infty,$$
$$\mathcal{N}_\infty = \{u \in H^1(\mathbb{R}^2) \setminus \{0\} : \ J'_\infty(u)u = 0\}$$
and
$$J_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |V|_\infty |u|^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx.$$

The main result related to above hypotheses is the following

Theorem 3. *Assume that V belongs to Class 1 or 2 and f verifies $(f_1) - (f_4)$. Then, problem $(P)_\epsilon$ has a positive solution for $\epsilon > 0$ small enough.*

In the proof of Theorem 3, we follow the ideas found in [9], because in that paper, problem $(P)_\epsilon$ has been considered with V verifying the same conditions, as explored in [6] and f satisfying the above assumptions.

Acknowledgments. The author is grateful to the referee for a number of helpful comments for improvement in this article.

References

[1] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14, (1973), 349-381.

[2] N. Ackermann and A. Szulkin, *A Concentration Phenomenon for Semilinear Elliptic Equations.* Arch. Ration. Mech. Anal. 207 (2013), 1075-1089.

[3] C.O. Alves, *Existence of positive solution for a nonlinear elliptic equation with saddle-like potential and nonlinearity with exponential critical growth in $\mathbb{R}^2$*, arXiv:1506.04947[math.AP]

[4] C.O. Alves, J.M. B. do Ó and M.A.S. Souto, *Local mountain-pass for a class of elliptic problems involving critical growth.* Nonlinear Anal. 46 (2001), 495-510.

[5] T. Bartsch, A. Pankov and Z.-Q. Wang, *Nonlinear Schrödinger equations with steep potential well*, Communications in Contemporany Mathematics, 3 (2001), 1-21.

[6] M. del Pino and P.L. Felmer, *Local Mountain Pass for semilinear elliptic problems in unbounded domains.* Calc. Var. Partial Differential Equations 4 (1996), 121-137.

[7] M. del Pino, P.L. Felmer and O.H. Miyagaki, *Existence of positive bound states of nonlinear Schrödinger equations with saddle-like potential*, Nonlinear Anal. 34 (1998), 979-989.

[8] M. del Pino and P.L. Felmer, *Semi-classical States for Nonlinear Schrödinger equations*, J. Funct. Anal. 149 (1997), 245-265.

[9] J. M. B. do Ó and M.A.S. Souto, *On a class of nonlinear Schrödinger equations in $\mathbb{R}^2$ involving critical growth*, J. Differential Equations 174 (2001), 289-311.

[10] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equations with bounded potential,* J. Funct. Anal. 69 (1986), 397-408.

[11] Y.G. Oh, *Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials on the class $(V)_a$,* Comm. Partial Differential Equations 13 (1988), 1499-1519.

[12] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. 43 (1992), 270-291

[13] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations,* Comm. Math. Physical 53 (1993), 229-244. .

Universidade Federal de Campina Grande, Unidade Acadêmica de Matemática, CEP:58429-900, Campina Grande - PB, Brazil.

E-mail address: `coalves@dme.ufcg.edu.br`

JEPE Vol 1, 2015, p. 243-262

Orthogonal $\perp$

A GLOBAL EXISTENCE RESULT FOR A KELLER-SEGEL TYPE SYSTEM WITH SUPERCRITICAL INITIAL DATA

DANIELE BARTOLUCCI(†), DANIELE CASTORINA(‡)

Abstract. We consider a parabolic-elliptic Keller-Segel type system, which is related to a simplified model of chemotaxis. Concerning the maximal range of existence of solutions, there are essentially two kinds of results: either global existence in time for general subcritical ($\|\rho_0\|_1 < 8\pi$) initial data, or blow–up in finite time for suitably chosen supercritical ($\|\rho_0\|_1 > 8\pi$) initial data with concentration around finitely many points. As a matter of fact there are no results claiming the existence of global solutions in the supercritical case. We solve this problem here and prove that, for a particular set of initial data which share large supercritical masses, the corresponding solution is global and uniformly bounded.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^2$ be any smooth and bounded domain, we consider the following parabolic-elliptic Keller-Segel type system

$$
\begin{cases}
\rho_t = \nabla \cdot (\nabla \rho - \rho \nabla (u + \log V)), & x \in \Omega, t > 0 \\
-\Delta u = \rho, & x \in \Omega, t \geq 0 \\
\rho(x,0) = \rho_0(x) \geq 0, \quad \int_\Omega \rho_0 = \lambda & x \in \Omega \\
\frac{\partial \rho}{\partial \nu} - \rho \frac{\partial (u + \log V)}{\partial \nu} = 0, \quad u = 0 & x \in \partial\Omega, t > 0
\end{cases}
\qquad P(\lambda, \Omega) \tag{1.1}
$$

We assume that V satisfies,

$$
V \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}),\ \alpha \in (0,1] \text{ and } 0 < a \leq V(x) \leq b. \tag{1.2}
$$

This is a simplified version of a chemotaxis model first introduced in [13]. The analysis of these kind of problems has attracted a lot of attention in recent years and we refer the interested reader to the monograph [19] for a complete account about this topic.

We say that (ρ, u) is a classical solution of $P(\lambda, \Omega)$ in $[0, T]$ if $\rho \geq 0$,

$$
\rho \in C^0(\overline{\Omega} \times [0,T]) \cap C^{2,1}\left(\overline{\Omega} \times (0,T]\right), \quad u \in C^{2,1}\left(\overline{\Omega} \times (0,T]\right), \tag{1.3}
$$

2010 *Mathematics Subject Classification.* 35J61, 35K45, 35K57, 35K58.

Key words and phrases. Keller-Segel; Chemotaxis; Global solution; Supercritical problems.
Received 01/07/2015, accepted 03/11/2015.
(†)Research partially supported by FIRB project *Analysis and Beyond* and by MIUR project *Metodi variazionali e PDE non lineari*.
(‡)Research partially supported by project *Bando Giovani Studiosi 2013 - Università di Padova - GRIC131695*.

and (ρ, u) solves (1.1). We say that (ρ, u) is a global solution of $P(\lambda, \Omega)$ if it is a classical solution in $[0, T]$ for any $T > 0$.

Concerning the maximal range (in time) of existence of solutions to $P(\lambda, \Omega)$, there are essentially two kinds of results. The first one yields sufficient conditions to guarantee that the solution is global and uniformly bounded, that is

$$\sup_{t>0} \sup_{x \in \overline{\Omega}} \rho(x,t) \leq C. \tag{1.4}$$

In fact it is well known that if ρ_0 is smooth and $\lambda < 8\pi$ then $P(\lambda, \Omega)$ admits a unique solution which is global and uniformly bounded, see [20] and also [6], [11]. The second class of results is about sufficient conditions which guarantee that blow up occurs in finite time, that is, there exist $T_{\max} > 0$ such that

$$\lim_{t \nearrow T_{\max}^-} \sup_{x \in \overline{\Omega}} \rho(x,t) = +\infty. \tag{1.5}$$

These conditions require the initial density ρ_0 to satisfy $\lambda > 8\pi$ and to be "peaked" around some point, see [18], [20] and also [12]. Other intermediate situations may occur, such as for example blow up in infinite time when $\lambda = 8\pi$, see [20]. So the value $\lambda = 8\pi$ is said to be the "critical" threshold and the study of (1.1) is generally divided in the subcritical and supercritical regime according to whether $\lambda < 8\pi$ or $\lambda > 8\pi$ respectively. Of course, there are many other results which are concerned with the blow up rate and the structure of the blow up set, see [19] and more recently [20] for further details.
The existence of global solutions in supercritical regimes is a challenging open problem in the study of general evolutionary equations, which has lately attracted a lot of attention, see [4, 5, 15]. Actually, in spite of a huge literature regarding the Keller-Segel model, there are no results at hand claiming the existence of global solutions to (1.1) in the supercritical case $\lambda > 8\pi$. We solve this problem here and prove that (1.1) may admit global and uniformly bounded solutions in the supercritical case as well. In fact we are able to find a particular set of initial data which share arbitrarily large supercritical masses $\lambda > 8\pi$ such that the corresponding solution is global and uniformly bounded. More exactly we have the following global existence result for (1.1).

Theorem 1. (a) *Let Ω be any smooth, bounded and simply connected domain. For any $c \in (0,1]$ let $D = \frac{a}{b}$ (as given in* (1.2)*) and $c_D = cD$. Then there exist $\overline{\epsilon}_* > \underline{\epsilon}_*(c_D) > 0$ such that if $\{\epsilon^2 x^2 + y^2 \leq \beta_-^2\} \subset \Omega \subset \{\epsilon^2 x^2 + y^2 \leq \beta_+^2\}$ with $c = \frac{\beta_-^2}{\beta_+^2}$ then, for any $\epsilon \in (0, \underline{\epsilon}_*(c_D)]$ and for any $\lambda \leq \lambda_{\epsilon, c_D}$, there exist initial data ρ_0 such that $P(\lambda, \Omega)$ admits a unique global and uniformly bounded solution $(\rho_\lambda, u_\lambda)$. Here $\underline{\lambda}_{\epsilon, c_D} < \lambda_{\epsilon, c_D} < \overline{\lambda}_\epsilon$ and $\underline{\lambda}_{\epsilon, c_D}$, $\overline{\lambda}_\epsilon$ are strictly decreasing (as functions of ϵ) in $(0, \underline{\epsilon}_*(c_D)]$, $(0, \overline{\epsilon}_*]$ respectively with $\underline{\lambda}_{\epsilon, c_D} \simeq \frac{4\pi c_D}{(8 - c_D)\epsilon}$, $\overline{\lambda}_\epsilon \simeq \frac{2\pi}{3\epsilon}$ as $\epsilon \to 0^+$.*

(b) *There exists $\bar{N} > 4\pi$ such that if Ω is any open, bounded and convex set whose isoperimetric ratio, $N \equiv N(\Omega) = \frac{L^2(\partial\Omega)}{A(\Omega)}$, satisfies $N \geq \bar{N}$, then for any $\lambda \leq \lambda_N$ there exist initial data ρ_0 such that $P(\lambda, \Omega)$ admits a unique global and uniformly bounded solution*

$(\rho_\lambda, u_\lambda)$. *Here* $\underline{\Lambda}_{N,\mathrm{D}} < \lambda_N < \overline{\Lambda}_N$ *and* $\underline{\Lambda}_{N,\mathrm{D}}$, $\overline{\Lambda}_N$ *are strictly increasing in* N *and* $\underline{\Lambda}_{N,\mathrm{D}} \simeq \frac{D\pi^2 N}{16(32-D)} + \mathrm{O}(1)$, $\overline{\Lambda}_N \simeq \frac{2\sqrt{3}N}{\pi} + \mathrm{O}(1)$ *as* $N \to +\infty$.

The proof of Theorem 1 is based on the following observation. System (1.1) admits a natural Lyapunov functional, which is the free energy (2.2) below. Suppose that we were able to find a strict local free energy minimizer, say $\rho_{0,\lambda}$. Then, in a carefully defined dual topology, those solutions of (1.1) with initial data in a small enough neighbourhood of $\rho_{0,\lambda}$ should be trapped there for any $t > 0$.

This information should yield the uniform estimates needed to prove global existence as well as uniform bounds. So the problem is to find out such minimizers and a good topology to work with. We seek these kind of minimizers in the class of stationary states (ρ, u) of (1.1) which therefore satisfy

$$\begin{cases} -\Delta\rho = \nabla\cdot(\rho\nabla(u+\log V)), & x\in\Omega \\ -\Delta u = \rho, & x\in\Omega \\ \int_\Omega \rho = \lambda & \\ \frac{\partial\rho}{\partial\nu} - \rho\frac{\partial(u+\log V)}{\partial\nu} = 0, \quad u = 0 & x\in\partial\Omega \end{cases} \tag{1.6}$$

We choose a particular steady state $(\rho_{0,\lambda}, u_\lambda)$ of the form $\rho_{0,\lambda} = \lambda\frac{Ve^{u_\lambda}}{\int_\Omega Ve^{u_\lambda}}$ (which satisfies the Neumann type boundary condition in (1.6) automatically) and so reduce the problem to the existence for large λ of a free energy minimizer in the form of a (possibly weak) solution u_λ of the following mean field equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u_\lambda = \lambda\dfrac{Ve^{u_\lambda}}{\int_\Omega Ve^{u_\lambda}} & \text{in } \Omega \\ u_\lambda = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1.7}$$

The existence of free energy minimizers taking the form $\lambda\frac{Ve^{u_\lambda}}{\int_\Omega Ve^{u_\lambda}}$ with $\lambda < 8\pi$ is well known [8] where u_λ is a minimizer of the corresponding variational functional, see (2.5) below. On the contrary, in case $\lambda > 8\pi$ solutions of (1.7) on domains with non trivial topology are well known to exist [9] which are not minimizers of (2.5) in general. The reason behind this issue is that if $\lambda > 8\pi$ then both the variational functional (2.5) and the free energy (2.2) are not bounded from below. Therefore, in particular, any local minimizer for $\lambda > 8\pi$ won't be a global one and would correspond not to a stable state but in fact to a so called metastable state from the dynamical point of view. Luckily enough, the existence for large λ of solutions of (1.7) on narrow domains which minimize the functional (2.5) has been recently established in [3] in case V is constant and then generalized to the case of non constant V for Liouville systems in [2]. In [2] it has also been shown that these solutions naturally yield minimizers of the multidimensional analogue of the free energy (2.2) in a suitable dual Orlicz-type topology. However we face a more subtle problem here since some properties which are almost obvious in the $H^1_0(\Omega)$-topology (see (2.7)) become more delicate in the Orlicz setting. As a consequence some care is needed to show that local minimizers of the free energy inherits that property from local minimizers of (2.5) in a suitable large enough sub cone, see in particular (3.18) in Proposition 2 below. To make the exposition self contained we will also provide the details of the existence result, see Theorem 2. This might be also useful since it clarifies the role played by a non constant V as well as the minimizing properties of those solutions in the scalar case.

Once we have found the desired minimizers with the necessary topological informations, then we can prove that in fact some crucial a priori bounds hold, see (4.2) below. By using these a priori estimates, then the proof of the global existence result could follow in principle from an adaptation of well known arguments [6], [11] to the analysis of system (1.1). It turns out that we do not need to work out this argument since this kind of adaptation has been recently worked out in [20] via a beautiful shorter proof of uniform boundedness and global existence of solutions based on (4.2).

Remark 1. It is likely that, by using a uniqueness result obtained in [3], one could also prove that the ρ_λ found in Theorem 1 converge as $t \to +\infty$ to the pair $(\rho_{0,\lambda}, u_\lambda)$ solving (1.7).

This paper is organized as follows. In section 2 we recall some well known facts about Keller-Segel systems. In section 4 we prove a global existence result for (1.1). In section 3 we obtain the needed free energy minimizers with large masses. Finally, in section 5, we prove Theorem 1.

2. Preliminaries

We collect here some well known results, see [19]. Some proofs are provided for reader's convenience.
The fact that for a given smooth initial value ρ_0, (1.1) is locally well posed in time is well known (see for example [6], [11]). Also, the fact that $\rho(\cdot, t) \geq 0$ follows for non negative initial value ρ_0 by the maximum principle. Actually it holds $\rho(\cdot, t) > 0$ for any $t > 0$, and in particular solutions are classical, see for example Theorem 3.1 in [19]. The total mass conservation for (1.1) can be obtained as follows:

$$\frac{d}{dt}\int_\Omega \rho\, dx = \int_\Omega \nabla\cdot(\nabla\rho - \rho\nabla(u + \log V))\, dx = \int_{\partial\Omega}\left(\frac{\partial\rho}{\partial\nu} - \rho\frac{\partial(u+\log V)}{\partial\nu}\right) d\sigma = 0. \tag{2.1}$$

Let us then consider the free energy associated to problem (1.1), that is

$$\mathcal{F}(\rho) = \int_\Omega \rho\left(\log\left(\frac{\rho}{V}\right) - 1\right) - \frac{1}{2}\int_\Omega \rho\, G[\rho] = \int_\Omega \rho\left(\log\left(\frac{\rho}{V}\right) - 1 - \frac{u}{2}\right), \tag{2.2}$$

where G is the Green function of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions, $G[\rho](x) = (G * \rho)(x)$ and $\rho \in \mathcal{P}_\lambda$ where,

$$\mathcal{P}_\lambda = \left\{\rho \in L^1(\Omega) : \rho \geq 0 \text{ a.e. }, \int_\Omega \rho = \lambda \text{ and } \int_\Omega \rho\log\rho < \infty\right\}. \tag{2.3}$$

Clearly, for any $t > 0$ for which $\rho(x,t)$ is defined, we have $\rho \in \mathcal{P}_\lambda$ whenever ρ_0 is (say) smooth, see for example [19]. Thanks to the conservation of mass (2.1) we can deduce an important property of $P(\lambda, \Omega)$: the decrease of the free energy $\mathcal{F}$ along the flow associated with the first equation in (1.1). In fact, notice that by (1.1) and integration by parts we have

$$\int_\Omega \rho u_t = -\int_\Omega u_t \Delta u = \int_\Omega \nabla u_t \nabla u = \int_\Omega u\rho_t.$$

From this and the fact that V is time independent, the homogeneous Neumann conditions and again integrating by parts, we deduce that:

$$\frac{d}{dt}\mathcal{F}(\rho) = \frac{d}{dt}\int_\Omega \rho\left(\log\left(\frac{\rho}{V}\right) - 1 - \frac{u}{2}\right) dx =$$
$$= \int_\Omega \rho_t\left(\log\left(\frac{\rho}{V}\right) - \frac{u}{2}\right) dx - \int_\Omega \rho_t\, dx + \int_\Omega \rho\frac{d}{dt}\left(\log\left(\frac{\rho}{V}\right) - \frac{u}{2}\right) dx =$$
$$= \int_\Omega \rho_t\left(\log\left(\frac{\rho}{V}\right) - \frac{u}{2}\right) dx + \int_\Omega \left(\rho_t - \frac{1}{2}\rho u_t\right) dx =$$
$$= \int_\Omega \rho_t\left(\log\left(\frac{\rho}{V}\right) - u\right) dx =$$
$$= \int_\Omega \nabla\cdot(\nabla\rho - \rho\nabla(u+\log V))\left(\log\left(\frac{\rho}{V}\right) - u\right) dx =$$
$$= \int_{\partial\Omega}\left(\frac{\partial\rho}{\partial\nu} - \rho\frac{\partial(u+\log V)}{\partial\nu}\right)\left(\log\left(\frac{\rho}{V}\right) - u\right) d\sigma$$
$$- \int_\Omega (\nabla\rho - \rho\nabla(u+\log V))\nabla\left(\log\left(\frac{\rho}{V}\right) - u\right) dx =$$
$$- \int_\Omega \rho\left|\nabla\left(\log\left(\frac{\rho}{V}\right) - u\right)\right|^2 dx \leq 0.$$

Hence we get that:

$$\mathcal{F}(\rho(\cdot,t)) \leq \mathcal{F}(\rho(\cdot,0)) = \mathcal{F}(\rho_0(\cdot)) \quad \forall\, t>0. \tag{2.4}$$

For any $v \in H_0^1(\Omega)$ let us consider the variational functional

$$J_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \lambda\log\left(\int_\Omega V e^v\right). \tag{2.5}$$

It is well known that critical points u_λ of J_λ are weak solutions of (1.7). As mentioned in the introduction we will need the following recently derived [2] existence result of strict local minimizers for J_λ with large mass.

Remark 2. If $v \in H_0^1(\Omega)$ then we define

$$\|v\|_{H_0^1(\Omega)} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}.$$

Theorem 2. (a) *Let Ω be any smooth and simply connected domain. For any $c \in (0,1]$ let $D = \frac{a}{b}$ (as given in (1.2)) and $c_D = cD$. Then there exist $\overline{\epsilon}_* > \underline{\epsilon}_*(c_D) > 0$ such that if $\{\epsilon^2x^2 + y^2 \leq \beta_-^2\} \subset \Omega \subset \{\epsilon^2x^2 + y^2 \leq \beta_+^2\}$ with $c = \frac{\beta_-^2}{\beta_+^2}$ then, for any $\epsilon \in (0, \underline{\epsilon}_*(c_D)]$ and for any $\lambda \leq \lambda_{\epsilon,c_D}$, problem (1.7) admits a solution u_λ. Here $\underline{\lambda}_{\epsilon,c_D} < \lambda_{\epsilon,c_D} < \overline{\lambda}_\epsilon$ and $\underline{\lambda}_{\epsilon,c_D}$, $\overline{\lambda}_\epsilon$ are strictly decreasing (as functions of ϵ) in $(0,\underline{\epsilon}_*(c_D)]$, $(0,\overline{\epsilon}_*]$ respectively with $\underline{\lambda}_{\underline{\epsilon}_*(c_D),c_D} = 8\pi = \overline{\lambda}_{\overline{\epsilon}_*}$ and $\underline{\lambda}_{\epsilon,c_D} \simeq \frac{4\pi c_D}{(8-c_D)\epsilon}$, $\overline{\lambda}_\epsilon \simeq \frac{2\pi}{3\epsilon}$ as $\epsilon \to 0^+$.*

(b) *There exists $\bar{N} > 4\pi$ such that if Ω is an open, bounded and convex set (therefore simple) whose isoperimetric ratio, $N \equiv N(\Omega) = \frac{L^2(\partial\Omega)}{A(\Omega)}$, satisfies $N \geq \bar{N}$, then for any $\lambda \leq \lambda_N$*

problem (1.7) admits a solution u_λ. *Here* $\underline{\Lambda}_{N,\mathrm{D}} < \lambda_N < \overline{\Lambda}_N$ *with* $\underline{\Lambda}_{\bar{N},\mathrm{D}} = 8\pi$, $\underline{\Lambda}_{N,\mathrm{D}}$ *and* $\overline{\Lambda}_N$ *strictly increasing in* N *and* $\underline{\Lambda}_{N,\mathrm{D}} \simeq \frac{D\pi^2 N}{16(32-D)} + \mathrm{O}(1)$, $\overline{\Lambda}_N \simeq \frac{2\sqrt{3}N}{\pi} + \mathrm{O}(1)$ *as* $N \to +\infty$.

(c) *The solution* u_λ *found in both cases* (a) *and* (b) *is a strict local minimizer of* J_λ *and the first eigenvalue of the linearized problem for (1.7) at* u_λ *is strictly positive. In particular, there exists* $\delta_0 > 0$ *such that*

$$J_\lambda(u) > J_\lambda(u_\lambda) \tag{2.6}$$

for any $0 < \|u - u_\lambda\|_{H_0^1(\Omega)} < \delta_0$. *Moreover, there exists* $\eta_0 > 0$ *such that, for any* $0 < \eta_1 < \eta_0$, *there exists* $d_1 > 0$ *such that*

$$J_\lambda(u) \geq J_\lambda(u_\lambda) + d_1 \tag{2.7}$$

whenever $\eta_1 \leq \|u - u_\lambda\|_{L^2(\Omega)} < \eta_0$.

Proof. We shall derive part (a) of the statement and skip the details of the proof of part (b) which can be handled by the same argument adopted in [3].
In view of the dilation invariance of (1.7) (that is, if $u_\lambda(z)$ solves (1.7) in Ω, then for any $t > 0$ $u_\lambda(tz)$ solves (1.7) in $(t)^{-1}\Omega$ with $V(z)$ replaced by $V(tz)$ which still satisfies (1.2)) for fixed $c \in (0,1]$ and up to a rescaling we can assume without loss of generality that

$$\Omega_{\epsilon,c} := \{\epsilon^2 x^2 + y^2 \leq c\} \subseteq \Omega \subseteq \{\epsilon^2 x^2 + y^2 \leq 1\} =: \Omega_\epsilon.$$

Let us consider the following Liouville-type [16] problem

$$\begin{cases} -\Delta u = \mu V e^u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.8}$$

and let us define

$$v_{\epsilon,\gamma} = 2\log\left(\frac{(1+\gamma^2)}{1+\gamma^2(\epsilon^2 x^2 + y^2)}\right), \quad (x,y) \in \Omega_\epsilon. \tag{2.9}$$

A straightforward evaluation shows that $v_{\epsilon,\gamma}$ satisfies

$$\begin{cases} -\Delta v_{\epsilon,\gamma} = V_{\epsilon,\gamma} e^{v_{\epsilon,\gamma}} & \text{in} \quad \Omega_\epsilon \\ v_{\epsilon,\gamma} = 0 & \text{on} \quad \partial\Omega_\epsilon, \end{cases} \tag{2.10}$$

where

$$V_{\epsilon,\gamma}(x,y) = \frac{4\gamma^2}{(1+\gamma^2)^2}\left(1 + \epsilon^2 + \gamma^2(1-\epsilon^2)(\epsilon^2 x^2 - y^2)\right). \tag{2.11}$$

Since

$$V_{\epsilon,\gamma}(x,y) \geq g_+(\gamma,\epsilon) := \frac{4\gamma^2}{(1+\gamma^2)^2}\left(1 + \epsilon^2 + \gamma^2(\epsilon^2 - 1)\right), \quad \forall (x,y) \in \Omega_\epsilon,$$

thanks to (1.2) we easily verify that $v_{\epsilon,\gamma}$ is a supersolution of (2.8) whenever

$$b\mu \leq g_+(\gamma,\epsilon). \tag{2.12}$$

For fixed $\epsilon \in (0,1)$, the function $h_\epsilon(t) = g_+(\sqrt{t},\epsilon)$ satisfies $h_\epsilon(0) = 0 = h_\epsilon\left(\frac{1+\epsilon^2}{1-\epsilon^2}\right)$, is strictly increasing in $\left(0, \frac{1+\epsilon^2}{3-\epsilon^2}\right)$ and strictly decreasing in $\left(\frac{1+\epsilon^2}{3-\epsilon^2}, \frac{1+\epsilon^2}{1-\epsilon^2}\right)$.

Therefore, putting $\overline{\gamma}_\epsilon^2 = \frac{1+\epsilon^2}{3-\epsilon^2}$ and $\overline{\mu}_{\epsilon,b} := \frac{1}{b} h_\epsilon\left(\overline{\gamma}_\epsilon^2\right) \equiv \frac{1}{b} g_+(\overline{\gamma}_\epsilon, \epsilon) \equiv \frac{(\epsilon^2+1)^2}{2b}$, we see in particular that for each $\mu \in (0, \overline{\mu}_{\epsilon,b}]$ there exists a unique $\gamma_\epsilon^+ \in (0, \overline{\gamma}_\epsilon]$ such that $\frac{1}{b} g_+(\gamma_\epsilon^+, \epsilon) = \mu$ and $v_{\epsilon,\gamma_\epsilon^+}$ is a supersolution of (2.8). Indeed we have

$$\left(\gamma_\epsilon^+\right)^2 = \left(\gamma_\epsilon^+(\mu)\right)^2 = \frac{2(1+\epsilon^2) - \mu b - 2\sqrt{(1+\epsilon^2)^2 - 2\mu b}}{\mu b + 4(1-\epsilon^2)}.$$

On the other hand let us consider

$$v_{\epsilon,\gamma,c} = \begin{cases} 2\log\left(\dfrac{(1+\gamma^2)}{1+\frac{\gamma^2}{c}(\epsilon^2 x^2 + y^2)}\right), & (x,y) \in \Omega_{\epsilon,c} \\ 0, & (x,y) \in \Omega \setminus \Omega_{\epsilon,c}. \end{cases} \tag{2.13}$$

Again a straightforward computation shows that $v_{\epsilon,\gamma,c}$ satisfies

$$\begin{cases} -\Delta v_{\epsilon,\gamma,c} = V_{\epsilon,\gamma,c} e^{v_{\epsilon,\gamma,c}} & \text{in} \quad \Omega_{\epsilon,c} \\ v_{\epsilon,\gamma,c} = 0 & \text{on} \quad \partial\Omega_{\epsilon,c}, \end{cases}$$

where

$$V_{\epsilon,\gamma,c}(x,y) = \begin{cases} \frac{4\gamma^2}{c(1+\gamma^2)^2}\left(1+\epsilon^2 + \frac{\gamma^2}{c}(1-\epsilon^2)(\epsilon^2 x^2 - y^2)\right) & \text{in } \Omega_{\epsilon,c} \\ 0 & \text{in } \Omega \setminus \Omega_{\epsilon,c}. \end{cases}$$

Since

$$V_{\epsilon,\gamma,c}(x,y) \le g_-(\gamma,\epsilon,c) := \frac{4\gamma^2}{c(1+\gamma^2)^2}\left(1+\epsilon^2+\gamma^2(1-\epsilon^2)\right), \quad \forall (x,y) \in \Omega,$$

it is not difficult to check that $v_{\epsilon,\gamma,c}$, again by (1.2), is a subsolution of (2.8) whenever

$$a\mu \ge g_-(\gamma,\epsilon,c). \tag{2.14}$$

For fixed $\epsilon \in (0,1)$, the function $f_{\epsilon,c}(t) = g_-(\sqrt{t},\epsilon,c)$, $t \in (0, \overline{\gamma}_\epsilon^2]$ is strictly increasing and satisfies $f_{\epsilon,c}(t) > h_\epsilon(t)$. Once again, putting $\overline{\gamma}_\epsilon^2 = \frac{1+\epsilon^2}{3-\epsilon^2}$ and $\underline{\mu}_{\epsilon,a} := \frac{1}{a} h_\epsilon\left(\overline{\gamma}_\epsilon^2\right) \equiv \frac{1}{a} g_-(\overline{\gamma}_\epsilon, \epsilon) \equiv \frac{(\epsilon^2+1)^2}{2a}$, we see that for each $\mu \in (0, \underline{\mu}_{\epsilon,a}]$ there exists a unique $\gamma_{\epsilon,c}^- \in (0, \overline{\gamma}_\epsilon)$ such that $\frac{1}{a} g_-(\gamma_{\epsilon,c}^-, \epsilon, c) = \mu$, $\gamma_{\epsilon,c}^- < \gamma_\epsilon^+$ and $v_{\epsilon,\gamma_{\epsilon,c}^-,c}$ is a subsolution of (2.8). Indeed we have

$$\left(\gamma_{\epsilon,c}^-\right)^2 = \left(\gamma_{\epsilon,c}^-(\mu)\right)^2 = \frac{\mu ac - 2(1+\epsilon^2) + 2\sqrt{(1+\epsilon^2)^2 - 2\epsilon^2\mu ac}}{4(1-\epsilon^2) - \mu ac}.$$

Notice that since $a < b$ we clearly have $\underline{\mu}_{\epsilon,a} > \overline{\mu}_{\epsilon,b}$. In conclusion, since $\gamma_{\epsilon,c}^-(\mu) \le \gamma_\epsilon^+(\mu)$ implies $v_{\epsilon,\gamma_{\epsilon,c}^-,c} \le v_{\epsilon,\gamma_\epsilon^+}$, for fixed $\epsilon \in (0,1)$ and for each $\mu \in (0, \overline{\mu}_{\epsilon,b}]$ we can set

$$\underline{u}_\mu = v_{\epsilon,\gamma_{\epsilon,c}^-(\mu),c}, \quad \overline{u}_\mu = v_{\epsilon,\gamma_\epsilon^+(\mu)},$$

and conclude via well known sub-supersolution results [10] that a solution (in a suitable weak sense) $u_{\epsilon,\mu,c}$ of (2.8) exists which satisfies

$$v_{\epsilon,\gamma_{\epsilon,c}^-(\mu),c} \le u_{\epsilon,\mu,c} \le v_{\epsilon,\gamma_\epsilon^+(\mu)}, \quad \forall (x,y) \in \Omega. \tag{2.15}$$

Then the Brezis-Merle results [7] and standard elliptic regularity follows that $u_{\epsilon,\mu,c}$ is a classical solution of (2.8).

Any such a solution $u_{\epsilon,\mu,c}$ therefore solves (1.7) with $\lambda = \lambda_{\epsilon,c_D}(\mu)$ satisfying

$$\lambda = \lambda_{\epsilon,c_D}(\mu) = \mu \int\limits_{\Omega} V e^{u_{\epsilon,\mu,c}} \geq \mu a \int\limits_{\Omega_{\epsilon,c}} e^{v_{\epsilon,\gamma^-_{\epsilon,c}(\mu),c}} = \mu a c \frac{\pi}{\epsilon} \left(1 + (\gamma^-_{\epsilon,c}(\mu))^2\right), \tag{2.16}$$

and

$$\lambda = \lambda_{\epsilon,c_D}(\mu) = \mu \int\limits_{\Omega} V e^{u_{\epsilon,\mu,c}} \leq \mu b \int\limits_{\Omega_{\epsilon}} e^{v_{\epsilon,\gamma^+_{\epsilon}(\mu)}} = \mu b \frac{\pi}{\epsilon} \left(1 + (\gamma^+_{\epsilon}(\mu))^2\right). \tag{2.17}$$

In case $\mu = \overline{\mu}_{\epsilon,b}$, recalling that $D = \frac{a}{b}$ and $c_D = cD$, we have

$$(\gamma^-_{\epsilon,c}(\overline{\mu}_{\epsilon}))^2 \equiv \underline{\gamma}^2_{\epsilon,c} = (1+\epsilon^2) \frac{c_D(1+\epsilon^2) + 4\left(\sqrt{1 - c_D\epsilon^2} - 1\right)}{8(1-\epsilon^2) - c_D(1+\epsilon^2)^2}$$

and

$$(\gamma^+_{\epsilon}(\overline{\mu}_{\epsilon}))^2 \equiv \overline{\gamma}^2_{\epsilon} = (1+\epsilon^2) \frac{3-\epsilon^2}{8(1-\epsilon^2) + (1+\epsilon^2)^2},$$

so that, by (2.16) and (2.17) respectively we have:

$$\lambda_{\epsilon,c_D} := \lambda_{\epsilon,c_D}(\overline{\mu}_{\epsilon,b}) \geq \underline{\lambda}_{\epsilon,c_D} = \frac{ac(1+\epsilon^2)^2}{2b} \frac{\pi}{\epsilon}(1 + \underline{\gamma}^2_{\epsilon,c}) \simeq \frac{4\pi c_D}{(8-c_D)\epsilon}, \tag{2.18}$$

and

$$\lambda_{\epsilon,c_D} \leq \overline{\lambda}_{\epsilon} = \frac{(1+\epsilon^2)^2}{2} \frac{\pi}{\epsilon}(1 + \overline{\gamma}^2_{\epsilon}) \simeq \frac{2\pi}{3\epsilon} \tag{2.19}$$

as $\epsilon \to 0^+$. Moreover it is easy to verify that $\underline{\lambda}_{\epsilon,c_D}$ is strictly decreasing at least for $\epsilon \in (0, \frac{1}{2\sqrt{10}}]$ and that there exists $\underline{\epsilon}_*(c_D) < \frac{1}{2\sqrt{10}}$ such that $\underline{\lambda}_{\epsilon,c_D} \geq 8\pi$ for any $\epsilon \in (0, \underline{\epsilon}_*(c_D)]$. These estimates are uniform in $0 < D \leq 1$ and in $c \in (0,1]$. We also see that $\overline{\lambda}_{\epsilon} \to \left(\frac{4\pi}{b}\right)^-$ as $\epsilon \to 1^-$, is strictly decreasing for $\epsilon \in (0, \epsilon_p]$ and strictly increasing for $\epsilon \in [\epsilon_p, 1)$ for some $\epsilon_p \simeq 0.5$ and then it is straightforward to check that there exists $\overline{\epsilon}_* > \underline{\epsilon}_*(c)$ such that $\overline{\lambda}_{\epsilon} \geq 8\pi$ for any $\epsilon \in (0, \overline{\epsilon}_*]$. Finally, since $\lambda_{\epsilon,c_D}(\mu)$ is continuous in μ and by using (2.16) and (2.17)

$$0 < \lambda_{\epsilon,c_D}(\mu) \leq \mu \frac{\pi}{\epsilon} \left(1 + (\gamma^+_{\epsilon}(\mu))^2\right) \stackrel{\text{as } \mu \to 0}{\longrightarrow} 0,$$

we obtain the existence of a solution for $P(\lambda, \Omega)$ not only for $\lambda = \lambda_{\epsilon,c_D}$, but for any $\lambda \in (0, \lambda_{\epsilon,c_D}]$ as well. This fact concludes the proof of the existence result claimed in (a).

Next we prove that the solutions obtained so far are strict local minimizers of J_λ, i.e. part (c) of the statement. Actually we need a stronger result, that is, the linearized problem relative to (1.7) has a strictly positive first eigenvalue. Putting

$$\omega = \omega(u) = \frac{Ve^u}{\int\limits_{\Omega} Ve^u}, \quad \text{and} \quad < f >_{\omega} = \int\limits_{\Omega} \omega(u) f,$$

then the linearized problem for (1.7) takes the form

$$\begin{cases} -\Delta\varphi - \lambda\omega(u)\varphi + \lambda\omega(u) < \varphi >_{\omega} = 0 & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Letting $H \equiv H_0^1(\Omega)$ and

$$\mathcal{L}(\phi,\psi)=\int\limits_{\Omega}(\nabla\phi\cdot\nabla\psi)-\lambda\int\limits_{\Omega}\omega(u)\phi\psi+\lambda\left(\int\limits_{\Omega}\omega(u)\phi\right)\left(\int\limits_{\Omega}\omega(u)\psi\right),\ (\phi,\psi)\in H\times H,$$

then by definition $\varphi \in H$ is a weak solution of (2.20) if

$$\mathcal{L}(\varphi,\psi)=0,\quad \forall\,\psi\in H.$$

We define $\tau \in \mathbb{R}$ to be an eigenvalue of the operator

$$L[\varphi]:=-\Delta\varphi-\lambda\omega(u)(\varphi-<\varphi>_{\omega}),\quad \varphi\in H,$$

if there exists a weak solution $\phi_0 \in H\setminus\{0\}$ of the linear problem

$$-\Delta\phi_0-\lambda\omega(u)\phi_0+\lambda\omega(u)<\phi_0>_{\omega}=\tau\omega(u)\phi_0\ \text{ in }\ \Omega, \tag{2.21}$$

that is, if

$$\mathcal{L}(\phi_0,\psi)=\tau\int\limits_{\Omega}\omega(u)\phi_0\psi,\quad \forall\,\psi\in H.$$

Standard arguments show that the eigenvalues form an unbounded (from above) sequence

$$\tau_1\le\tau_2\le\cdots\le\tau_n\cdots,$$

with finite dimensional eigenspaces (although the first eigenfunction changes sign and cannot be assumed to be simple in this situation).

Let us define

$$Q(\phi)=\frac{\mathcal{L}(\phi,\phi)}{<\phi^2>_{\omega}}=\frac{\int\limits_{\Omega}|\nabla\phi|^2-\lambda<\phi^2>_{\omega}+\lambda<\phi>_{\omega}^2}{<\phi^2>_{\omega}},\quad \phi\in H.$$

In particular it is not difficult to prove that the first eigenvalue can be characterized as follows

$$\tau_1=\inf\{Q(\phi)\,|\,\phi\in H\setminus\{0\}\}.$$

At this point we argue by contradiction and assume that $\tau_1\le 0$. Therefore we readily conclude that

$$\tau_0:=\inf\{Q_0(\phi)\,|\,\phi\in H\setminus\{0\}\}\le 0,\quad \text{where}\quad Q_0(\phi)=\frac{\mathcal{L}_0(\phi,\phi)}{<\phi^2>_{\omega}}$$

and

$$\mathcal{L}_0(\phi,\psi)=\int\limits_{\Omega}(\nabla\phi\cdot\nabla\psi)-\lambda\int\limits_{\Omega}\omega(u)\phi\psi,\ (\phi,\psi)\in H\times H.$$

Clearly τ_0 is attained by a simple and positive eigenfunction φ_0 which satisfies

$$\begin{cases}-\Delta\varphi_0-\lambda\omega(u)\varphi_0=\tau_0\omega(u)\varphi_0 & \text{in }\ \Omega\\ \varphi_0=0 & \text{on }\ \partial\Omega.\end{cases} \tag{2.22}$$

Let us recall that we have obtained solutions for (1.7) as solutions of (2.8) in the form $u=u_{\epsilon,\mu,c}$, for some $\mu=\mu(\epsilon)\le\overline{\mu}_{\epsilon,b}$ whose value of $\lambda=\lambda(\mu,\epsilon,c_D)$ was then estimated as a function of ϵ. That point of view is well suited for our purpose, that is, we get back to

$\mu = \lambda \left(\int_\Omega V e^u\right)^{-1}$. Hence, let us observe that for a generic value $\mu \leq \overline{\mu}_{\epsilon,b}$ (2.22) takes the form

$$\begin{cases} -\Delta\varphi_0 - \mu V K_{\epsilon,\mu,c}\varphi_0 = \nu_0 V K_{\epsilon,\mu,c}\varphi_0 & \text{in } \Omega \\ \varphi_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where

$$K_{\epsilon,\mu,c} = e^{u_{\epsilon,\mu,c}} \quad \text{and} \quad \nu_0 = \mu\frac{\tau_0}{\lambda} \leq 0.$$

We observe that, by defining

$$K^{(-)}_{\epsilon,\mu,c} := e^{v_{\epsilon,\gamma^-_{\epsilon,c}(\mu),c}} = \begin{cases} \left(\frac{1+\gamma^-_{\epsilon,c}(\mu)^2}{1+\frac{\gamma^-_{\epsilon,c}(\mu)^2}{c}(\epsilon^2x^2+y^2)}\right)^2 & (x,y) \in \Omega_{\epsilon,c} \\ 1 & (x,y) \in \Omega \setminus \Omega_{\epsilon,c}, \end{cases}$$

$$K^{(+)}_{\epsilon,\mu} := e^{v_{\epsilon,\gamma^+_\epsilon(\mu)}} = \left(\frac{1+\gamma^+_\epsilon(\mu)^2}{1+\gamma^+_\epsilon(\mu)^2(\epsilon^2x^2+y^2)}\right)^2, \qquad (x,y) \in \Omega_\epsilon$$

we have

$$K^{(-)}_{\epsilon,\mu,c} \leq K_{\epsilon,\mu,c} \leq K^{(+)}_{\epsilon,\mu} \qquad \text{for any } (x,y) \in \Omega.$$

In particular, since

$$K^{(+)}_{\epsilon,\mu} \leq (1+\gamma^+_\epsilon(\mu)^2)^2 \quad \text{and} \quad 1 \leq K^{(-)}_{\epsilon,\mu,c} \leq (1+\gamma^-_{\epsilon,c}(\mu)^2)^2 \quad \text{in} \quad \Omega,$$

and

$$\Omega \subset T_\epsilon := \{(x,y) \in \mathbb{R}^2 \,|\, |x| \leq (\epsilon)^{-1},\ |y| \leq 1\}, \tag{2.24}$$

then, by using the fact that

$$\nu_0 = \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2\,dx - \mu\int_\Omega K_{\epsilon,\mu}\varphi^2\,dx}{\int_\Omega K_{\epsilon,\mu}\varphi^2\,dx} \,\middle|\, \varphi \in H \right\} \leq 0,$$

it is not difficult to check that, for some $\mu \leq \overline{\mu}_{\epsilon,b} = \frac{(1+\epsilon^2)^2}{2b}$, thanks to (1.2) the following inequality holds:

$$\inf\left\{ \frac{\int_{T_\epsilon} |\nabla\varphi|^2\,dx - \mu b(1+\gamma^+_\epsilon(\mu)^2)^2\int_{T_\epsilon}\varphi^2\,dx}{\int_{T_\epsilon}\varphi^2\,dx} \,\middle|\, \varphi \in H \right\} \leq 0. \tag{2.25}$$

Hence, there exists $\overline{\mu}_0 \leq 0$ such that, putting $\sigma = \sigma(\mu,\epsilon) = \mu b(1+\gamma^+_\epsilon(\mu)^2)^2 + \overline{\mu}_0$, there exists a weak solution $\phi_0 \in H$ of

$$\begin{cases} -\Delta\phi_0 - \sigma\phi_0 = 0 & \text{in } T_\epsilon, \\ \phi_0 = 0 & \text{on } \partial T_\epsilon. \end{cases} \tag{2.26}$$

It is well known that the minimal eigenvalue σ_{min} of (2.26) satisfies $\sigma_{min} = \frac{\pi^2}{4}\epsilon^2 + \frac{\pi^2}{4} > 2(1+\epsilon^2)$ and we conclude that

$$2(1+\epsilon^2) \leq \sigma(\mu,\epsilon) = \mu b(1+\gamma^+_\epsilon(\mu)^2)^2 + \overline{\mu}_0. \tag{2.27}$$

Next, since $\underline{\epsilon}_*(c_D) < \frac{1}{2\sqrt{10}}$, it is not difficult to check that $\sigma = \sigma(\mu,\epsilon)$ satisfies

$$\sigma(\mu,\epsilon) \leq 1,$$

for any $\epsilon \leq \underline{\epsilon}_*(c_D)$, which is of course a contradiction to (2.27). Therefore τ_1 is strictly positive as claimed, and this clearly yields (2.6) for some small $\delta_0 > 0$.

Finally we prove (2.7): on the one hand, since $\|u - u_\lambda\|_{H^1_0(\Omega)} < \delta_0$, then by Poincaré inequality with constant $\mu_1 = \mu_1(\Omega)$ we have $\|u - u_\lambda\|_{L^2(\Omega)} < \eta_0 := \frac{\delta_0}{\mu_1}$. On the other hand, by a simple Taylor expansion for J_λ around the local minimizer u_λ and by choosing η_0 smaller if necessary, we see that

$$J_\lambda(u) = J_\lambda(u_\lambda) + \frac{1}{2} < J''_\lambda(u_\lambda)[u - u_\lambda], u - u_\lambda > + \mathrm{o}(\|u - u_\lambda\|^2_{L^2(\Omega)})$$
$$\geq J_\lambda(u_\lambda) + \frac{1}{4} < J''_\lambda(u_\lambda)[u - u_\lambda], u - u_\lambda >$$

for $\|u - u_\lambda\|_{L^2(\Omega)} < \eta_0$. Now, thanks to the fact that the linearized operator $\mathcal{L}$ admits a positive first eigenvalue τ_1 we have:

$$< J''_\lambda(u_\lambda)[u - u_\lambda], u - u_\lambda >= \mathcal{L}(u - u_\lambda, u - u_\lambda) \geq \tau_1 < (u - u_\lambda)^2 >_\omega$$

Notice that, thanks to (1.2) and (2.15), there exists $c_\lambda > 0$ such that $\omega(u_\lambda) \geq c_\lambda$, which spells that

$$< (u - u_\lambda)^2 >_\omega \geq c_\lambda \|u - u_\lambda\|_{L^2(\Omega)}$$

Thus, we finally find that for any $\eta_1 < \eta_0$ we have

$$J_\lambda(u) \geq J_\lambda(u_\lambda) + \frac{\tau_1 c_\lambda \eta_1}{4}$$

whenever $\eta_1 \leq \|u - u_\lambda\|_{L^2(\Omega)} < \eta_0$, which proves (2.7) with $d_1 := \frac{\tau_1 c_\lambda \eta_1}{4}$. □

3. The existence of free energy minimizers with large masses.

As mentioned in the introduction we will need some properties in the dual Orlicz topology that the free energy minimizers inherit from those of the u_λ found in Theorem 2. In order to clarify this aspect let

$$L_\Phi(\Omega) := \{\, \rho \,\text{measurable in}\ \Omega \,:\, \|\rho\|_\Phi < +\infty\},$$

be the Orlicz space [1], [14] whose Young functions are

$$\Phi(t) = t\log(1+t), \quad t \geq 0, \quad \Psi(s) = \max_{t\geq 0}\{ts - \Phi(t)\}, \quad s \geq 0,$$

where

$$\|\rho\|_\Phi := \sup_h \left\{ \left| \int_\Omega \rho h \right|, \int_\Omega \Psi(|h|) \leq 1 \right\}.$$

The Orlicz space contains the so called Orlicz class of all functions ρ which are measurable in Ω and satisfy $\int_\Omega \Phi(|\rho|) < +\infty$. For any u which satisfies $\int_\Omega \Psi(|u|) < +\infty$, let us also introduce the Luxemburg norm,

$$\|u\|_\Psi := \inf\left\{ \alpha > 0 \,:\, \int_\Omega \Psi(\alpha^{-1}|u|) \leq 1 \right\}. \tag{3.1}$$

It is straightforward to check that

$$\Psi(s) \leq s e^{s-1},\ s \in [0, +\infty) \quad \text{and} \quad \Psi(s) \leq e^{s-1} - (s-1),\ s \geq 2, \tag{3.2}$$

whence in particular the Moser-Trudinger inequality [17] shows that (3.1) is well defined for any $u \in H_0^1(\Omega)$. It is well known [1], [14] that $L_\Phi(\Omega)$ is a Banach space with respect to the $\|\cdot\|_\Phi$-norm. In particular the following version of the Hölder inequality holds

$$\int_\Omega \rho u \leq \|\rho\|_\Phi \|u\|_\Psi, \tag{3.3}$$

for any $\rho \in L_\Phi(\Omega)$ and any u such that (3.1) is well defined. In particular we have

$$\frac{1}{2}t^2 \leq \Phi(t) \leq t^2,\ t \leq 1, \quad \text{and} \quad t \leq \Phi(t) \leq t^2,\ t \geq 1. \tag{3.4}$$

Clearly $\mathcal{P}_\lambda$ is a convex subset of $L_\Phi(\Omega)$. We will need the following result about the continuity of $\mathcal{F}$ with respect to the topology induced by $L_\Phi(\Omega)$. Let

$$\mathcal{P} := \left\{ \rho \in L^1(\Omega) : \rho \geq 0 \ \text{a.e.} \quad \text{and} \quad \int_\Omega \rho \log \rho < \infty \right\}.$$

For any density $\rho \in \mathcal{P}$ we will let u_ρ be the corresponding potential, that is $u_\rho(x) = G[\rho](x) = (G * \rho)(x)$. Clearly u_ρ is the unique distributional solution of:

$$\begin{cases} -\Delta u_\rho = \rho & \text{in } \Omega \\ \quad u_\rho = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

Remark 3. By using the Green's representation formula

$$u_\rho(z) = \int_\Omega G(z,w)\rho(w)\,dw, \quad \forall\, z \in \Omega, \tag{3.6}$$

and the Hölder's inequality (3.3) we see that $u_\rho \in L^\infty(\Omega)$. Indeed, for any $\alpha \geq \frac{1}{2\pi}$ we have

$$\int_\Omega \Psi\left(\frac{|G(z,w)|}{\alpha}\right) = \int_\Omega \Psi\left(\frac{G(z,w)}{\alpha}\right) dw \leq \int_\Omega \left(\tfrac{G(z,w)}{\alpha}\right) e^{\frac{G(z,w)}{\alpha}}\,dw \leq$$

$$\frac{C_\Omega}{\alpha} \int\limits_{B_1(z)} \log\frac{1}{|z-w|}\left(\frac{1}{|z-w|}\right)^{\frac{1}{2\pi\alpha}} dw \leq \frac{C_\Omega}{\alpha} \int\limits_{B_1(z)} \log\frac{1}{|z-w|}\left(\frac{1}{|z-w|}\right) dw \leq \frac{C_0}{\alpha},$$

so that

$$\sup_{z\in\Omega} \|G(z,\cdot)\|_\Psi \leq C_0 \tag{3.7}$$

for some constant $C_0 \geq 1$ depending only by Ω. Therefore, in particular $\rho u_\rho \in L^1(\Omega)$ and then standard truncation arguments show that $u_\rho \in H_0^1(\Omega)$. We conclude that u_ρ is also a weak solution of (3.5).

Then we have

Proposition 1. *The functional $\mathcal{F}$ is continuous on $\mathcal{P}$ with respect to the topology induced by $L_\Phi(\Omega)$.*

Proof. Fix $\rho \in \mathcal{P}$ and let $\{\rho_n\} \subset \mathcal{P}$ be any sequence such that

$$\|\rho_n - \rho\|_\Phi \to 0, \text{ as } n \to +\infty. \tag{3.8}$$

We are going to prove that $\mathcal{F}(\rho_n) \to \mathcal{F}(\rho)$. We recall that, since Φ satisfies the following Δ_2-condition

$$\Phi(2t) \leq 4\Phi(t),\ \forall\, t \geq 1 \tag{3.9}$$

then, see [14], a sequence $\{\rho_n\} \subset L_\Phi(\Omega)$ satisfies (3.8) if and only if

$$\lim_{n\to+\infty} \int_\Omega \Phi(|\rho_n - \rho|) = 0. \tag{3.10}$$

Clearly ρ_n satisfies (3.10) since $\mathcal{P} \subset L_\Phi(\Omega)$. By using (3.4) we find

$$\int_\Omega |\rho_n - \rho| \leq \int\limits_{|\rho_n-\rho|\leq 1} |\rho_n - \rho| + \int\limits_{|\rho_n-\rho|\geq 1} |\rho_n - \rho| \leq$$

$$|\Omega|^{\frac{1}{2}} \left(\int\limits_{|\rho_n-\rho|\leq 1} |\rho_n - \rho|^2 \right)^{\frac{1}{2}} + \int\limits_{|\rho_n-\rho|\geq 1} |\rho_n - \rho| \leq$$

$$|\Omega|^{\frac{1}{2}} \left(\int_\Omega 2\Phi(\rho_n - \rho) \right)^{\frac{1}{2}} + \int_\Omega \Phi(\rho_n - \rho).$$

Therefore $\rho_n \to \rho$ in $L^1(\Omega)$. Next observe that

$$u_n(z) := G[\rho_n](z) = \int_\Omega G(z,w)\rho_n(w), \quad \text{and} \quad u_\rho(z) = G[\rho](z) = \int_\Omega G(z,w)\rho(w),$$

satisfy $u_n \to u_\rho$ in L^∞. In fact, letting z_n be any point where the maximum of $|u_n - u_\rho|$ is attained, we find

$$\|u_n - u_\rho\|_\infty = |u_n(z_n) - u_\rho(z_n)| \leq \int_\Omega G(z_n,w)|\rho_n(w) - \rho(w)| \leq \|\rho_n - \rho\|_\Phi \|G(z_n,\cdot)\|_\Psi \leq C_0\|\rho_n - \rho\|_\Phi,$$

where we used (3.3) and (3.7). Therefore, since ρ_n converges in $L^1(\Omega)$ to ρ, we conclude that

$$\int_\Omega \rho_n G[\rho_n] = \int_\Omega \rho_n u_n \to \int_\Omega \rho u_\rho = \int_\Omega \rho G[\rho], \text{as } \ n \to +\infty,$$

by the duality $L^1(\Omega), L^\infty(\Omega)$. This fact shows that the functional $\rho \mapsto \int_\Omega \rho G[\rho]$ is continuous.

Since we have shown that $\int_\Omega \rho G[\rho]$ is continuous and that $\rho_n \to \rho$ in $L^1(\Omega)$, then, to conclude the proof, we just need to show that in fact $\int_\Omega \rho_n \log(\rho_n) \to \int_\Omega \rho \log(\rho)$. To this aim we observe that since ρ_n satisfies (3.8), and since $L_\Phi(\Omega)$ is a Banach space, then $\|\rho_n\|_\Phi$ is uniformly bounded. But then, see [14] §3.10.9, since Φ satisfies (3.9), then there exists $C_2 > 0$ depending only by $\int_\Omega \rho \log \rho$ such that

$$\int_\Omega \Phi(\rho_n) \leq C_2, \ \forall\, n \in \mathbb{N},$$

whence in particular

$$\int_\Omega \rho_n \log(\rho_n) \leq \int\limits_{\rho_n \geq 1} \rho_n \log(\rho_n) \leq \int\limits_{\rho_n \geq 1} \rho_n \log(1+\rho_n) \leq C_2, \ \forall\, n \in \mathbb{N}. \tag{3.11}$$

At this point, for any $\varepsilon > 0$ we can choose $m_\varepsilon \in (0,1)$ such that, setting

$$\Omega_{m,1} := \{\rho_n \leq m\} \cup \{\rho \leq m\},$$

then, for any $m < m_\varepsilon$ it holds

$$\left|\int\limits_{\Omega_{m,1}} \rho_n \log(\rho_n) - \int\limits_{\Omega_{m,1}} \rho \log(\rho)\right| \leq \int\limits_{\Omega} 2m|\log(m)| < \varepsilon, \tag{3.12}$$

where we used the fact that $m\log(m) \to 0$ as $m \to 0^+$. Next let us set $\Omega_m := \Omega \setminus \Omega_{m,1}$ and decompose

$$\left|\int_\Omega \rho_n \log(\rho_n) - \int_\Omega \rho \log(\rho)\right| \leq \tag{3.13}$$

$$\left|\int\limits_{\Omega_{m,1}} \rho_n \log(\rho_n) - \int\limits_{\Omega_{m,1}} \rho \log(\rho)\right| + \left|\int\limits_{\Omega_m} \rho_n \log(\rho_n) - \int\limits_{\Omega_m} \rho \log(\rho)\right| \leq$$

$$2\varepsilon + \left|\int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} \rho_n \log(\rho_n) - \int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} \rho \log(\rho)\right| + \left|\int\limits_{\Omega_m \cap \{\rho_n \leq \rho\}} \rho_n \log(\rho_n) - \int\limits_{\Omega_m \cap \{\rho_n \leq \rho\}} \rho \log(\rho)\right|$$

We will use fact that for any $\alpha \geq 1$, we have

$$\int_\Omega \Psi\left(\frac{|\log(\rho_n)|\chi_{\Omega_m}}{\alpha}\right) \leq \int_\Omega \left(\frac{|\log(\rho_n)|\chi_{\Omega_m}}{\alpha}\right) e^{\frac{|\log(\rho_n)|\chi_{\Omega_m}}{\alpha}} =$$

$$\int\limits_{\Omega_m} \left(\frac{|\log(\rho_n)|}{\alpha}\right) e^{\frac{|\log(\rho_n)|}{\alpha}} \leq \int\limits_{m \leq \rho_n \leq 1} \left(\frac{|\log(\rho_n)|}{\alpha}\right) e^{\frac{|\log(\rho_n)|}{\alpha}} + \int\limits_{\rho_n \geq 1} \left(\frac{|\log(\rho_n)|}{\alpha}\right) e^{\frac{|\log(\rho_n)|}{\alpha}} \leq$$

$$|\Omega|\frac{|\log(m)|}{\alpha}\left(\frac{1}{m}\right)^{\frac{1}{\alpha}} + \int\limits_{\rho_n \geq 1} \frac{\log(\rho_n)}{\alpha}(\rho_n)^{\frac{1}{\alpha}} \leq |\Omega|\frac{|\log(m)|}{\alpha}\left(\frac{1}{m}\right) + \int\limits_{\rho_n \geq 1} \frac{\log(\rho_n)}{\alpha}(\rho_n) =$$

$$\frac{1}{\alpha}\left(\frac{|\Omega||\log(m)|}{m} + \int\limits_{\rho_n \geq 1} \rho_n \log(\rho_n)\right) \leq \frac{1}{\alpha}\left(\frac{|\Omega||\log(m)|}{m} + C_2\right),$$

showing that, in view of (3.11),

$$\|\log(\rho_n)\chi_{\Omega_m}\|_\Psi \leq C_*,$$

for some constant C_* depending only by m, $|\Omega|$ and C_2. Therefore we can estimate

$$\left|\int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} \rho_n \log(\rho_n) - \int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} \rho \log(\rho)\right| \leq \tag{3.14}$$

$$\int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} |\rho_n - \rho||\log(\rho_n)| + \int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} \rho|\log(\rho_n) - \log(\rho)| \leq$$

$$\|\rho_n - \rho\|_\Phi \|\log(\rho_n)\chi_{\Omega_m}\|_\Psi + \int\limits_{\Omega_m \cap \{\rho_n \geq \rho\}} \rho \frac{1}{\min\{\rho_n, \rho\}}|\rho_n - \rho| \leq$$

$$C_*\|\rho_n-\rho\|_\Phi+\int\limits_{\Omega_m\cap\{\rho_n\geq\rho\}}\rho\frac{1}{\rho}|\rho_n-\rho|\leq$$

$$C_*\|\rho_n-\rho\|_\Phi+\int\limits_{\Omega}|\rho_n-\rho|<\varepsilon,$$

for any n large enough, possibly depending on m and C_2, where we used the mean value theorem. The same argument in a slightly easier form shows that

$$\left|\int\limits_{\Omega_m\cap\{\rho_n\leq\rho\}}\rho_n\log(\rho_n)-\int\limits_{\Omega_m\cap\{\rho_n\leq\rho\}}\rho\log(\rho)\right|<\varepsilon \tag{3.15}$$

for any n large enough, possibly depending on m, and we skip the details relative to this estimate to avoid repetitions. The fact that $\int_\Omega\rho_n\log(\rho_n)\to\int_\Omega\rho\log(\rho)$ is an immediate consequence of (3.13), (3.14) and (3.15). $\square$

Next we have

Proposition 2. *Let u_λ be a strict local minimum of J_λ and assume that (2.6), (2.7) hold. If*

$$\rho_{0,\lambda}=\lambda\frac{Ve^{u_\lambda}}{\int_\Omega Ve^{u_\lambda}}, \tag{3.16}$$

then $\rho_{0,\lambda}\in\mathcal{P}_\lambda$ and the following property $(\mathbf{H})_\lambda$ holds: there exist and $\varepsilon_0>\varepsilon_1>0$ such that

$$\mathcal{F}(\rho)-\mathcal{F}(\rho_{0,\lambda})>0, \tag{3.17}$$

for any $\rho\in\mathcal{P}_\lambda$ such that $0<\|\rho-\rho_{0,\lambda}\|_\Phi<\varepsilon_0$ and

$$\mathcal{F}(\rho)\geq\mathcal{F}(\rho_{0,\lambda})+d_1, \tag{3.18}$$

for any $\rho\in L^2(\Omega)\cap\mathcal{P}_\lambda$ such that $\|\rho-\rho_{0,\lambda}\|_\Phi=\varepsilon_1$, where d_1 is the constant introduced in (2.7).

Proof. Whenever $u_\rho=G[\rho](x)=(G*\rho)(x)$ we set

$$\sigma_{u_\rho}:=\lambda\frac{Ve^{u_\rho}}{\int_\Omega Ve^{u_\rho}}. \tag{3.19}$$

If $\rho_{0,\lambda}$ is defined as in (3.16), then $\rho_{0,\lambda}\in\mathcal{P}_\lambda$ and for any $\rho\in\mathcal{P}_\lambda$ we find,

$$\mathcal{F}(\rho)-\mathcal{F}(\rho_\lambda)=$$

$$=\int_\Omega\rho\left(\log\left(\frac{\rho}{V}\right)-1-\frac{1}{2}G[\rho]\right)-\int_\Omega\rho_{0,\lambda}\left(\log\left(\frac{\rho_{0,\lambda}}{V}\right)-1-\frac{1}{2}G[\rho_{0,\lambda}]\right)=$$

$$=\int_\Omega\rho\left(\log\left(\frac{\rho}{V}\right)-\frac{1}{2}G[\rho]\right)+\lambda\log\left(\int_\Omega Ve^{u_\lambda}\right)-\frac{1}{2}\int_\Omega|\nabla u_\lambda|^2-\lambda\log\lambda-\int_\Omega\rho+\int_\Omega\rho_{0,\lambda}=$$

$$=\int_\Omega\rho\log\left(\frac{\rho}{\sigma_{u_\rho}}\right)+\int_\Omega\rho\log\left(\frac{\sigma_{u_\rho}}{V}\right)-\frac{1}{2}\int_\Omega\rho G[\rho]-\lambda\log\lambda-J_\lambda(u_\lambda)\geq$$

$$=\int_\Omega\rho\log\left(\frac{\sigma_{u_\rho}}{V}\right)-\frac{1}{2}\int_\Omega\rho G[\rho]-\lambda\log\lambda-J_\lambda(u_\lambda)=$$

$$= \int_\Omega \rho u_\rho - \frac{1}{2}\int_\Omega \rho G[\rho] - \lambda \log\left(\int_\Omega V e^{u_\rho}\right) - J_\lambda(u_\lambda) =$$

$$= J_\lambda(u_\rho) - J_\lambda(u_\lambda), \tag{3.20}$$

where we have used Remark 3, (3.19) and the following facts:

- by definition $\log\left(\frac{\rho_{0,\lambda}}{V}\right) = \log\left(\frac{\lambda e^{u_\lambda}}{\int_\Omega V e^{u_\lambda}}\right) = u_\lambda + \log\lambda - \log\left(\int_\Omega V e^{u_\lambda}\right)$;
- by (1.7) and the definition of $\rho_{0,\lambda}$ we have: $-\int_\Omega \rho_{0,\lambda} u_\lambda + \frac{1}{2}\int_\Omega \rho_{0,\lambda} G[\rho_{0,\lambda}] = -\frac{1}{2}\int_\Omega |\nabla u_\lambda|^2$;
- since $\{\sigma_{u_\rho}, \rho_{0,\lambda}, \rho\} \subset \mathcal{P}_\lambda$ we have $\int_\Omega \sigma_{u_\rho} = \int_\Omega \rho_{0,\lambda} = \int_\Omega \rho = \lambda$;
- by Jensen's inequality applied to $\phi(t) = t\log t$ and $t = \frac{\rho}{\sigma_{u_\rho}}$ we have: $\int_\Omega \rho \log\left(\frac{\rho}{\sigma_{u_\rho}}\right) \geq 0$.

We learned of this nice application of the Jensen's inequality in [21]. In view of (2.6) we have $J_\lambda(u) - J_\lambda(u_\lambda) > 0$ for any $0 < \|u - u_\lambda\|_{H_0^1(\Omega)} < \delta_0$. Therefore, to prove (3.17), it only remains to show that there exists $\varepsilon_0 = \varepsilon_0(\delta_0) > 0$ such that $0 < \|u_\rho - u_\lambda\|_{H_0^1(\Omega)} < \delta_0$ whenever $0 < \|\rho - \rho_{0,\lambda}\|_\Phi < \varepsilon_0$. We first prove a stronger property which will be needed in the proof of (3.18) as well. By using the Green's representation formula and the Hölder's inequality (3.3) we see that

$$\|u_\rho - u_\lambda\|_{L^\infty(\Omega)} \leq \sup_{x\in\Omega} \int_\Omega G(x,y)|\rho(y) - \rho_{0,\lambda}(y)|\,dy \leq C_0\|\rho - \rho_{0,\lambda}\|_\Phi \tag{3.21}$$

where C_0 is the constant found in (3.7). Next observe that

$$\int_\Omega |\nabla(u_\rho - u_\lambda)|^2 = \int_\Omega (|\nabla u_\rho|^2 - 2(\nabla u_\rho, \nabla u_\lambda) + |\nabla u_\lambda|^2) = \int_\Omega \rho(u_\rho - u_\lambda) + \int_\Omega u_\lambda(\rho_{0,\lambda} - \rho) \leq$$

$$C_0\|\rho - \rho_{0,\lambda}\|_\Phi + \|u_\lambda\|_\Psi \|\rho - \rho_{0,\lambda}\|_\Phi \leq C\|\rho - \rho_{0,\lambda}\|_\Phi,$$

where we used (3.2), (3.3), (3.21) and the fact that obviously $\|u_\lambda\|_\Psi$ is bounded. We conclude that there exists $C > 1$ such that

$$\|u_\rho - u_\lambda\|_{H_0^1(\Omega)} \leq C\|\rho - \rho_{0,\lambda}\|_\Phi, \tag{3.22}$$

and in particular that it is always possible to fix $\varepsilon_0 := \frac{\delta_0}{2C} > 0$ so that $0 \leq \|u_\rho - u_\lambda\|_{H_0^1(\Omega)} < \delta_0$ whenever $0 < \|\rho - \rho_{0,\lambda}\|_\Phi < \varepsilon_0$. Hence (3.17) follows whenever we can prove that if ρ has been chosen in this way, and therefore does not coincide with $\rho_{0,\lambda}$, then the unique u_ρ determined through (3.5) does not coincide with u_λ. However this is easily verified since if this was not the case we would find

$$0 = -\Delta(u_\rho - u_\lambda) = (\rho - \rho_{0,\lambda}) \quad \text{in } \Omega$$

which is in contradiction with the fact that ρ does not coincide with $\rho_{0,\lambda}$. At this point (3.20) shows that

$$\mathcal{F}(\rho) - \mathcal{F}(\rho_\lambda) \geq J_\lambda(u_\rho) - J_\lambda(u_\lambda) > 0,$$

whenever $0 < \|\rho - \rho_{0,\lambda}\|_\Phi < \varepsilon_0$ as claimed.

Concerning (3.18) we first observe that, in view of (3.4), we have $L^2(\Omega) \subset L_\Phi(\Omega)$. The linear operator $T : L^2(\Omega) \mapsto H_0^1(\Omega)$ which maps $\rho \in L^2(\Omega)$ to the unique weak solution $u_\rho \in H_0^1(\Omega)$ of (3.5) is a continuous bijection, whence there exists $C_3 > 0$ such that

$$\|\rho\|_{L^2(\Omega)} \leq C_3\|u_\rho\|_{H_0^1(\Omega)}, \quad \forall\, \rho \in L^2(\Omega),$$

and then, since $\rho_{0,\lambda} \in L^2(\Omega)$, for any $\rho \in L^2(\Omega) \cap \mathcal{P}$ we find

$$\|\rho - \rho_{0,\lambda}\|_{L^2(\Omega)} \le C_3 \|u_\rho - u_\lambda\|_{H_0^1(\Omega)}. \tag{3.23}$$

On the other side (3.4) implies

$$\int_\Omega \Phi(|\rho - \rho_{0,\lambda}|) \le \int_\Omega |\rho - \rho_{0,\lambda}|^2, \tag{3.24}$$

and then we have

Lemma 1. *For any $\varepsilon_1 < \varepsilon_0$ there exists $\delta_2 > 0$ such that*

$$\|\rho - \rho_{0,\lambda}\|_{L^2(\Omega)} \ge \delta_2, \tag{3.25}$$

for any $\rho \in L^2(\Omega) \cap \mathcal{P}$ such that $\|\rho - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$.

Proof. If the claim were false we could find a sequence ρ_n such that $\|\rho_n - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$ and $\|\rho_n - \rho_{0,\lambda}\|_{L^2(\Omega)} \le \frac{1}{n}$, as $n \to +\infty$. Then (3.24) implies that $\int_\Omega \Phi(|\rho_n - \rho_{0,\lambda}|) \to 0$, as $n \to +\infty$ which in view of (3.10) is the same as (3.8), that is $\|\rho_n - \rho_{0,\lambda}\|_\Phi \to 0$, as $n \to +\infty$. This is the desired contradiction to $\|\rho_n - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$ □

At this point, by using (3.22), (3.23) and (3.25) we conclude that for any $\varepsilon_1 < \varepsilon_0$ we have

$$\frac{\delta_2}{C_3} \le \|u_\rho - u_\lambda\|_{H_0^1(\Omega)} \le C\varepsilon_1,$$

for any $\rho \in L^2(\Omega) \cap \mathcal{P}$ such that $\|\rho - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$. Then, in particular, we can choose $\varepsilon_1 < \frac{\delta_0}{2C}$ and a smaller δ_2 if needed which satisfies $\delta_{0,1} := \frac{\delta_2}{C_3} < C\varepsilon_1$ to conclude that

$$\delta_{0,1} \le \|u_\rho - u_\lambda\|_{H_0^1(\Omega)} < \delta_0, \tag{3.26}$$

for any $\rho \in L^2(\Omega) \cap \mathcal{P}$ such that $\|\rho - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$. At this point we can prove the following

Claim: *There exists $0 < \eta_1 < \eta_0 := \delta_0$ such that*

$$\eta_1 \le \|u_\rho - u_\lambda\|_{L^2(\Omega)} < \eta_0,$$

for any $\rho \in L^2(\Omega) \cap \mathcal{P}$ such that $\|\rho - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$.

Proof of Claim:
Clearly (3.26) holds and then in particular we see that $\|u_\rho - u_\lambda\|_{L^2(\Omega)} < \delta_0$. Concerning the left hand side inequality we argue by contradiction. If the claim were false we could find a sequence ρ_n such that, setting $u_n = u_{\rho_n}$, then, in view of (3.25), we would find

$$\|u_n - u_\lambda\|_{H_0^1(\Omega)} \ge \delta_{0,1} \quad \text{and} \quad \|u_n - u_\lambda\|_{L^2(\Omega)} \le \frac{1}{n}, \ \forall\, n \in \mathbb{N}. \tag{3.27}$$

On the other side we have $-\Delta(u_n - u_\lambda) = (\rho_n - \rho_\lambda)$ and then, multiplying by $u_n - u_\lambda$ and integrating by parts, we find

$$\|\nabla(u_n - u_\lambda)\|^2_{L^2(\Omega)} = \int_\Omega (\rho_n - \rho_{0,\lambda})(u_n - u_\lambda) \le \|\rho_n - \rho_{0,\lambda}\|_{L^2(\Omega)} \|u_n - u_\lambda\|_{L^2(\Omega)} \le$$

$$C_3 \|u_n - u_\lambda\|_{H_0^1(\Omega)} \frac{1}{n} \le C_3 C \|\nabla(u_n - u_\lambda)\|_{L^2(\Omega)} \frac{1}{n}, \ \forall\, n \in \mathbb{N},$$

where we used (3.23) and the Sobolev's inequality to conclude that $\|u_n - u_\lambda\|_{H_0^1(\Omega)} \leq C\|\nabla(u_n - u_\lambda)\|_{L^2(\Omega)}$, for some uniform constant $C > 0$. Therefore we would have $\|\nabla(u_n - u_\lambda)\|^2_{L^2(\Omega)} \to 0$ and then in particular $\|u_n - u_\lambda\|_{H_0^1(\Omega)} \to 0$, as $n \to +\infty$, which is a contradiction to (3.27). □

By using the statement of the Claim, then (3.20) and (2.7) imply that (3.18) holds. □

4. A global existence result for (1.1)

With the aid of property $(\mathbf{H})_\lambda$ in Proposition 2 a global existence result follows by a standard Lyapunov stability argument to be applied to a set of suitably chosen initial data.

Proposition 3. *Suppose that $(\mathbf{H})_\lambda$ in Proposition 2 holds for some $\lambda > 0$. Then there exists $\varepsilon_\lambda > 0$ such that if ρ_0 in (1.1) is any smooth and non negative density such that $\|\rho_0 - \rho_{0,\lambda}\|_\Phi \leq \varepsilon_\lambda$, then $\frac{\lambda}{2} \leq \int_\Omega \rho_0 \leq 2\lambda$ and the corresponding solution $(\rho_\lambda(\cdot,t), u_\lambda(\cdot,t))$ is global and uniformly bounded.*

Proof. Since $\mathcal{F}$ is continuous, we can choose $0 < \varepsilon_2 < \varepsilon_1$ such that

$$\mathcal{F}(\rho) \leq \mathcal{F}(\rho_{0,\lambda}) + \frac{d_1}{2},$$

for any $\|\rho - \rho_{0,\lambda}\|_\Phi \leq \varepsilon_2$. By taking a smaller value of ε_2 we may assume that $\frac{\lambda}{2} \leq \int_\Omega \rho \leq 2\lambda$ whenever $\|\rho - \rho_{0,\lambda}\|_\Phi \leq \varepsilon_2$. At this point let us choose ρ_0 in (1.1) to be any smooth and non negative density such that $\|\rho_0 - \rho_{0,\lambda}\|_\Phi \leq \varepsilon_2$ and let $(\rho_\lambda, u_\lambda)$ denote the corresponding unique solution in $[0,T]$, for some $T > 0$. Then, we claim that for any $t \in [0,T]$ it holds

$$\mathcal{F}(\rho_{0,\lambda}) \leq \mathcal{F}(\rho(\cdot,t)) < \mathcal{F}(\rho_{0,\lambda}) + d_1. \tag{4.1}$$

In fact, on one side the right hand inequality is always satisfied since (2.4) implies that

$$\mathcal{F}(\rho(\cdot,t)) \leq \mathcal{F}(\rho_0) \leq \mathcal{F}(\rho_{0,\lambda}) + \frac{d_1}{2}.$$

On the other side we also have $\rho(\cdot,t) \in \mathcal{P}_\lambda$ in view of the mass conservation and the fact that $\rho(x,t)$ is classical whence it satisfies (1.3) whenever it exists (see [19] Theorem 3.1). So, if for some $t_* > 0$ we would find that $\rho_* := \rho(x,t_*)$ satisfies $\mathcal{F}(\rho_*) < \mathcal{F}(\rho_{0,\lambda})$, then, in view of $(\mathbf{H})_\lambda$, necessarily $\|\rho_* - \rho_{0,\lambda}\|_\Phi \geq \varepsilon_0$. Therefore in particular, by the continuity of the norm, there existed $t_1 < t_*$ such that $\rho_1 := \rho(x,t_1)$ satisfied $\|\rho_1 - \rho_{0,\lambda}\|_\Phi = \varepsilon_1$. But then (3.18) implies that $\mathcal{F}(\rho_1) \geq \mathcal{F}(\rho_{0,\lambda}) + d_1$, since ρ_1 being classical surely belongs to $L^2(\Omega)$. This is the desired contradiction and thus we have proved that (4.1) holds. In particular it is easy to see that (4.1) implies that there exist $C > 0$, depending only on λ and d_1, such that

$$\int_\Omega \rho_\lambda(\cdot,t)(\log(\rho_\lambda(\cdot,t)) - 1) \leq C, \qquad \int_\Omega |\nabla u_\lambda(\cdot,t)|^2 \leq C, \tag{4.2}$$

for any $t \in [0,T]$. At this point we can follow step by step the argument in Theorem 3 in [20] to conclude that

$$\sup_{t\in[0,T]} \sup_{x\in\overline{\Omega}} \rho_\lambda(x,t) \leq \tilde{C},$$

for some uniform constant $\tilde{C} > 0$. Then well known arguments imply that the solution is global and the desired conclusion follows by choosing $\varepsilon_\lambda = \varepsilon_2$. $\square$

5. The Proof of Theorem 1

In this section we prove Theorem 1.

The Proof of Theorem 1

We discuss the proof of part (a). The proof of part (b) can be worked exactly with the same argument with minor changes.

By Theorem 2 we have a strict local minimizer u_λ of J_λ for any $\lambda \leq \lambda_{\epsilon,c_D}$, where $\underline{\lambda}_{\epsilon,c_D} < \lambda_{\epsilon,c_D} < \overline{\lambda}_\epsilon$ and $\underline{\lambda}_{\epsilon,c_D} \simeq \frac{4\pi c_D}{(8-c_D)\epsilon}$, $\overline{\lambda}_\epsilon \simeq \frac{2\pi}{3\epsilon}$ as $\epsilon \to 0^+$. It follows from Proposition 2 that if $\rho_{0,\lambda}$ is defined as in (3.16), then $\rho_{0,\lambda} \in \mathcal{P}_\lambda$ is a strict local minimizer of $\mathcal{F}$. Actually we have a stronger result since $(\mathbf{H})_\lambda$ in Proposition 2 holds. Therefore we can apply Proposition 3 to conclude that there exists $\varepsilon_\lambda > 0$ such that if ρ_0 in (1.1) is any smooth and non negative density such that $\|\rho_0 - \rho_{0,\lambda}\|_\Phi \leq \varepsilon_\lambda$, then $\frac{\lambda}{2} \leq \int_\Omega \rho_0 \leq 2\lambda$ and the corresponding solution $(\rho_\lambda, u_\lambda)$ is global and uniformly bounded. Let $2m_\lambda = \min\limits_{\overline{\Omega}} \rho_{0,\lambda}$. Clearly $m_\lambda > 0$ and we define f_λ to be any smooth function in $\overline{\Omega}$ which satisfies $|f_\lambda| \leq m_\lambda$ and $\int_\Omega f_\lambda = 0$. Then we can choose $0 < \sigma < \frac{1}{2}$ depending on ε_λ such that $\rho_0 = \rho_{0,\lambda} + \sigma f_\lambda$ satisfies $0 < \|\rho_0 - \rho_{0,\lambda}\|_\Phi \leq \varepsilon_\lambda$. Clearly $\int_\Omega \rho_0 = \lambda$ and then in particular $\rho_0 \in \mathcal{P}_\lambda$. Therefore, for any $\lambda \leq \lambda_{\epsilon,c_D}$ we have found initial data ρ_0 such that the solution $(\rho_\lambda, u_\lambda)$ of $P(\lambda,\Omega)$ is global and uniformly bounded as claimed. $\square$

References

[1] R.A. Adams, "Sobolev Spaces", Academic Press, New York, 1975.

[2] D. Bartolucci, *Existence and non existence results for supercritical systems of Liouville-type equations on simply connected domains*, Calc. Var. P. D. E., to appear DOI 10.1007/s00526-014-0750-9.

[3] D. Bartolucci, F. De Marchis, *Supercritical Mean Field Equations on convex domains and the Onsager's statistical description of two-dimensional turbulence*, Arch. Rat. Mech. Anal., **217**/2 (2015), 525-570; DOI: 10.1007/s00205-014-0836-8.

[4] J. Burczak, R. Granero-Belinchon, *Critical Keller- Segel meets Burgers on S1*, preprint arXiv:1504.00955.

[5] J. Burczak, R. Granero-Belinchon, *Global solutions for a supercritical drift-diffusion equation*, preprint arXiv:1507.00694.

[6] P. Biler, *Local and global solvability of some systems modelling chemotaxis*, Adv. Math. Sci. Appl. **8** (1998), 715-743.

[7] H. Brezis & F. Merle, *Uniform estimates and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. in P.D.E. **16**(8,9) (1991), 1223-1253.

[8] E. Caglioti, P.L. Lions, C. Marchioro & M. Pulvirenti, *A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description. II*, Comm. Math. Phys. **174** (1995), 229–260.

[9] C. C. Chen, C.S. Lin, *Topological Degree for a mean field equation on Riemann surface*, Comm. Pure Appl. Math. **56** (2003), 1667–1727.

[10] P. Clément & G. Sweers, *Getting a solution between sub- and supersolutions without monotone iteration*, Rend. Istit. Mat. Univ. Trieste **19** (1987), 189-194.

[11] H. Gajewski, K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. **195** (1998), 77-114.

[12] W. Jager, S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819-824.

[13] E.F. Keller, L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. **26** (1970), 399-415.
[14] A. Kufner, O. Jhon & S. Fučik, "Function spaces", Academia, Prague, 1977.
[15] A. Kiselev, X. Xu, *Suppression of chemotactic explosion by mixing*, preprint arXiv:1508.05333.
[16] J. Liouville, *Sur L' Équation aux Différence Partielles* $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$, C.R. Acad. Sci. Paris **36** (1853), 71–72.
[17] J. Moser, *A sharp form of an inequality by N.Trudinger*, Indiana Univ. Math. J. **20** (1971), 1077–1091.
[18] T. Senba, T. Suzuki, *Weak solutions to a parabolic-elliptic system of chemotaxis*, J. Funct. Anal. **191** (2002), 17-51.
[19] T. Suzuki, "Free Energy and Self-Interacting Particles", PNLDE **62**, Birkhauser, Boston, (2005).
[20] T. Suzuki, *Exclusion of boundary blowup for 2D chemotaxis system provided with Dirichlet boundary condition for the Poisson part*, J. Math. Pure Appl. **100** (2013), 347-367.
[21] G. Wolansky, *On the evolution of self-interacting clusters and applications to semilinear equations with exponential nonlinearity*, J. Anal. Math. **59** (1992), 251–272.

Daniele Bartolucci, Department of Mathematics, University of Rome *"Tor Vergata"*,, Via della ricerca scientifica n.1, 00133 Roma, Italy.
E-mail address: bartoluc@mat.uniroma2.it

Daniele Castorina, Dipartimento di Matematica, Università di Padova,, Via Trieste 63, 35121 Padova, Italy.
E-mail address: castorin@math.unipd.it

JEPE Vol 1, 2015, p. 263-269

Orthogonal ⊥

ON A NONLINEAR SYSTEM OF PDE'S ARISING IN FREE CONVECTION

BERNARD BRIGHI AND SENOUSSI GUESMIA

Abstract. By employing a fixed point argument, we prove existence and uniqueness of the weak solution to a nonlinear system of PDE's arising in free convection. An adapted weak formulation is involved to let the solution as weak as possible and avoid any additional smoothness hypotheses.

1. Introduction

In this paper, we consider a model problem introduced in [1], and derived from a coupled system of partial differential equations arising in the study of free convection about a vertical flat plate embedded in a porous medium. In [1], some existence result has been obtained for small data, by using the inverse function theorem. In [4], under more satisfying hypotheses, the existence of a solution is obtained by an iterative method. The regularity of this solution is also studied.

Here, we come back to the weak formulation of this problem and, under reasonable hypotheses on the data, we prove by constructing a suitable contraction mapping, that there is one and only one weak solution.

Details about the physical background can be found, for example, in [5], [6], [7], [8], [9] and [10]. In these papers, the authors assume that convection takes place in a thin layer around the plate. This allows to make boundary-layer approximations, and to get similarity solutions by solving an ordinary differential equation of the type

$$f''' + ff'' + \mathbf{g}(f') = 0$$

on the half line $[0, +\infty)$, with the boundary conditions $f(0) = a$, $f'(0) = b$ (or $f''(0) = c$) and $f'(t) \to \ell$ as $t \to +\infty$, where ℓ is a root of the function $\mathbf{g}$. This boundary value problem has been widely studied, most of the time for some particular form of $\mathbf{g}$, but also in the general case. The well known Blasius problem is for $\mathbf{g} = 0$. For an overview of mathematical results about these problems, we refer to [2] and [3] and the reference therein.

Let us state now the problem we are interesting in. Let Ω be a bounded domain of $\mathbb{R}^2$ with sufficiently smooth boundary Γ. Let Γ_1 and Γ_2 be two parts of Γ, such that $meas\,(\Gamma_1) \neq 0$ and

$$\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma, \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

2010 *Mathematics Subject Classification.* Primary: 35D30, 35J65, 47H09; Secondary: 76R10.

Key words and phrases. Existence and uniqueness, nonlinear elliptic system, contraction, fix point theorem, free convection.

Received 29/06/15, accepted 22/12/15.

In Ω, we consider the boundary value system defined by

$$-\Delta\Psi + K.\nabla H = F \tag{1.1}$$

$$-\lambda\Delta H + \nabla H.(\nabla\Psi)^{\perp} + \nabla\Theta.(\nabla\Psi)^{\perp} = 0 \tag{1.2}$$

with mixed boundary conditions for Ψ

$$\Psi = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial\Psi}{\partial\nu} = 0 \text{ on } \Gamma_2, \tag{1.3}$$

and for H

$$H = 0 \text{ on } \Gamma, \tag{1.4}$$

where $\overrightarrow{\nu}$ is the unit outward normal vector on Γ and $(\nabla\Psi)^{\perp} = (\partial_y\Psi, -\partial_x\Psi)$. The unknown functions are the stream function Ψ and the temperature H. The functions F, Θ and K are given, and we suppose that

$$F \in L^2(\Omega)$$

and that the function Θ belongs to $H^2(\Omega)$ and satisfies

$$\Delta\Theta = 0 \text{ in } \Omega. \tag{1.5}$$

Let us notice that $H^2(\Omega) \subset C^0(\overline{\Omega})$. For the coefficients $K = (k_1, k_2)$, it is assumed that

$$K \in L^{\infty}(\Omega) \times L^{\infty}(\Omega). \tag{1.6}$$

See [1] for details about the derivation of the problem (1.1)-(1.4). In the following, we will denote by $(\cdot,\cdot)$ the $L^2(\Omega)$-scalar product, and by $\|\cdot\|$ (resp. $|\cdot|_2$, $|\cdot|_{\infty}$ and $|\cdot|_{\infty,\Gamma}$) the norm of $H^1(\Omega)$ (resp. $L^2(\Omega)$, $L^{\infty}(\Omega)$ and $L^{\infty}(\Gamma)$). We also denote by $H_0^1(\Omega,\Gamma_1)$ the subset of functions of $H^1(\Omega)$ that vanish on Γ_1.

2. Weak formulation

In order to define a variational formulation of the previous problem, let us assume that Ψ and H are classical solutions of (1.1) and (1.2) in Ω, such that the boundary conditions (1.3) and (1.4) hold. Multiplying (1.1) and (1.2) by test functions $u \in H_0^1(\Omega,\Gamma_1)$ and $v \in H_0^1(\Omega)$, and integrating on Ω, we get

$$\int_{\Omega} \nabla\Psi.\nabla u\, dx + \int_{\Omega} u\, K.\nabla H dx = \int_{\Omega} F u\, dx$$

and

$$\lambda\int_{\Omega} \nabla H.\nabla v\, dx + \int_{\Omega} v\, \nabla H.(\nabla\Psi)^{\perp} dx + \int_{\Omega} v\, \nabla\Theta.(\nabla\Psi)^{\perp} dx = 0. \tag{2.1}$$

Since the system is formulated in the above form we are more able to think about solutions $\Psi \in H_0^1(\Omega,\Gamma_1)$ and $H \in H_0^1(\Omega)$. If this is the case, the third integral in the latter equality is still well defined (this is due to the fact that $\Theta \in H^2(\Omega)$), whereas, a priori, it is not anymore the case for the second one.

Let us clarify this point. To this end, for $u, v, w \in H^1(\Omega)$ such that $u\nabla v.(\nabla w)^{\perp} \in L^1(\Omega)$, we set

$$a(u,v,w) = \int_{\Omega} u\nabla v.(\nabla w)^{\perp} dx = \left(u\nabla v, (\nabla w)^{\perp}\right)$$

and let us show the following results.

Lemma 1. *Let $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ such that one of them vanishes on the boundary of Ω. For $w \in H^1(\Omega)$ we have*

$$a(u, v, w) = -a(v, u, w).$$

In particular, for every $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and every $w \in H^1(\Omega)$ we have : $a(u, u, w) = 0$.

Proof. For $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$ and $w \in H^1(\Omega)$ the quantities $a(u, v, w)$ and $a(v, u, w)$ are well defined. Since, moreover, $uv \in H_0^1(\Omega)$, we have

$$\begin{aligned} a(u, v, w) + a(v, u, w) &= \big(u\nabla v + v\nabla u, (\nabla w)^\perp\big) = \big(\nabla(uv), (\nabla w)^\perp\big) \\ &= -\big(\operatorname{div}\big((\nabla w)^\perp\big), uv\big)_{H^{-1}(\Omega), H_0^1(\Omega)} = 0, \end{aligned}$$

because $\operatorname{div}\big((\nabla w)^\perp\big) = 0$. □

Lemma 2. *Let $u, v \in H_0^1(\Omega)$. For $w \in H^2(\Omega)$ we have*

$$a(u, v, w) = -a(v, u, w).$$

In particular, for every $u \in H_0^1(\Omega)$ and every $w \in H^2(\Omega)$ we have : $a(u, u, w) = 0$.

Proof. First, because $H^1(\Omega) \hookrightarrow L^4(\Omega)$, the quantities $a(u, v, w)$ and $a(v, u, w)$ are well defined for all $u, v \in H_0^1(\Omega)$ and $w \in H^2(\Omega)$. On the other hand, by Lemma 1, for all $\varphi, \psi \in \mathcal{D}(\Omega)$ and all $w \in H^2(\Omega)$, we have

$$a(\varphi, \psi, w) = -a(\psi, \varphi, w),$$

and the conclusion then follows from the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$. □

Taking into account Lemma 1, we can replace the second integral in (2.1) by

$$-\int_\Omega H\nabla v.\,(\nabla\Psi)^\perp\,dx$$

which is well defined, if $H \in L^\infty(\Omega)$. Having that in mind, we state the following (equivalent) definition.

Definition 1. We will say that a couple (Ψ, H) is a WEAK SOLUTION of the problem (1.1)-(1.4), if $\Psi \in H_0^1(\Omega, \Gamma_1)$ and $H \in L^\infty(\Omega) \cap H_0^1(\Omega)$, and if the integral identities

$$\begin{aligned} (\nabla\Psi, \nabla u) + (K.\nabla H, u) &= (F, u) \\ \lambda(\nabla H, \nabla v) - a(H, v, \Psi) + a(v, \Theta, \Psi) &= 0 \end{aligned}$$

hold for any $u \in H_0^1(\Omega, \Gamma_1)$ and for any $v \in H_0^1(\Omega)$.

3. Existence and uniqueness of a weak solution

3.1. **A priory estimates.** We will need the following lemma.

Lemma 3. *Let $\Psi \in H^2(\Omega)$. If $H \in H_0^1(\Omega)$ satisfies*

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla H, \nabla v) - a(H, v, \Psi) = -a(v, \Theta, \Psi), \tag{3.1}$$

then

$$\inf_\Gamma \Theta \le H + \Theta \le \sup_\Gamma \Theta \tag{3.2}$$

and

$$\inf_\Gamma \Theta - \sup_\Omega \Theta \le H \le \sup_\Gamma \Theta - \inf_\Omega \Theta. \tag{3.3}$$

In particular, we have $H \in L^\infty(\Omega)$ and $|H|_\infty$ is bounded independently of Ψ.

Proof. The ingredients of the proof are in [1, Proposition 3.2] ; for convenience, and because the hypotheses are slightly different, we write it here. Let us set $l = \sup_\Gamma \Theta$ and $H^+ = \sup\{H + \Theta - l\,; 0\}$. Since $H^+ \in H^1_0(\Omega)$, then (1.5), (3.1) and Lemma 2 imply

$$\begin{aligned}\lambda(\nabla H^+, \nabla H^+) &= \lambda(\nabla H, \nabla H^+) + \lambda(\nabla \Theta, \nabla H^+) = \lambda(\nabla H, \nabla H^+)\\ &= a(H, H^+, \Psi) - a(H^+, \Theta, \Psi)\\ &= a(H, H^+, \Psi) + a(\Theta, H^+, \Psi)\\ &= a(H + \Theta, H^+, \Psi) = a(H^+, H^+, \Psi) = 0.\end{aligned}$$

It follows that $|\nabla H^+|_2 = 0$ and hence $H^+ = 0$. This gives the second inequality of (3.2). To obtain the other one, we set $l' = \inf_\Gamma \Theta$ and $H^- = \inf\{H + \Theta - l'\,; 0\}$ and proceed in the same way. The inequalities (3.3) follow immediately from (3.2). □

3.2. **A contraction.** Let $\mathbf{W} = H^1_0(\Omega, \Gamma_1) \times H^1_0(\Omega)$. On the Hilbert space $\mathbf{W}$ we define the norm $\| \; \|_{\mathbf{W}}$ by

$$\|(\Psi, H)\|_{\mathbf{W}} = \kappa |\nabla \Psi|_2 + |\nabla H|_2$$

where $\kappa > 0$ is a constant that we will choose later.

Let $\mathbf{D} = \mathcal{D}(\Omega, \Gamma_1) \times \mathcal{D}(\Omega)$ where $\mathcal{D}(\Omega, \Gamma_1)$ is the subset of $\mathcal{D}(\bar\Omega)$ whose elements are supported far away from Γ_1 and let $\mathbf{F} : \mathbf{D} \to \mathbf{W}$ be the application defined in the following way. If $(\Psi, H) \in \mathbf{D}$, then $\mathbf{F}(\Psi, H) = (\tilde\Psi, \tilde H)$ where $\tilde\Psi$ and $\tilde H$ are the unique solutions of the linear problems

$$\forall u \in H^1_0(\Omega, \Gamma_1), \quad (\nabla \tilde\Psi, \nabla u) = -(K.\nabla H, u) + (F, u), \tag{3.4}$$

$$\forall v \in H^1_0(\Omega), \quad \lambda(\nabla \tilde H, \nabla v) - a(\tilde H, v, \Psi) = -a(v, \Theta, \Psi). \tag{3.5}$$

Let us notice that the coercivity on $H^1_0(\Omega)$ of the bilinear form

$$(\tilde H, v) \mapsto \lambda(\nabla \tilde H, \nabla v) - a(\tilde H, v, \Psi)$$

follows from Lemma 2. If now $(\Psi_1, H_1) \in \mathbf{D}$ and $(\Psi_2, H_2) \in \mathbf{D}$, then we deduce from (3.4) and (3.5) that, for all $u \in H^1_0(\Omega, \Gamma_1)$ and for all $v \in H^1_0(\Omega)$, we have

$$(\nabla(\tilde\Psi_1 - \tilde\Psi_2), \nabla u) = -(K.\nabla(H_1 - H_2), u), \tag{3.6}$$

$$\lambda(\nabla(\tilde H_1 - \tilde H_2), \nabla v) - a(\tilde H_1, v, \Psi_1) + a(\tilde H_2, v, \Psi_2) = -a(v, \Theta, \Psi_1 - \Psi_2). \tag{3.7}$$

where $(\tilde\Psi_i, \tilde H_i) = \mathbf{F}(\Psi_i, H_i)$, $i = 1, 2$. Let us choose $u = \tilde\Psi_1 - \tilde\Psi_2$. On one hand, thanks to (1.6), we get from (3.6)

$$|\nabla(\tilde\Psi_1 - \tilde\Psi_2)|_2^2 = -(K.\nabla(H_1 - H_2), \tilde\Psi_1 - \tilde\Psi_2) \le |K|_\infty |\nabla(H_1 - H_2)|_2 |\tilde\Psi_1 - \tilde\Psi_2|_2,$$

and hence

$$|\nabla(\tilde\Psi_1 - \tilde\Psi_2)|_2 \le C|K|_\infty |\nabla(H_1 - H_2)|_2 \tag{3.8}$$

where C is the the Poincaré constant of Ω. On the other hand, using Lemma 1, we can rewrite (3.7), taking $v = \tilde{H}_1 - \tilde{H}_2$, as

$$\begin{aligned}
\lambda|\nabla(\tilde{H}_1 - \tilde{H}_2)|_2^2 &= a(\tilde{H}_1, \tilde{H}_1 - \tilde{H}_2, \Psi_1) - a(\tilde{H}_2, \tilde{H}_1 - \tilde{H}_2, \Psi_2) - a(\tilde{H}_1 - \tilde{H}_2, \Theta, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2, \tilde{H}_1, \Psi_1) - a(\tilde{H}_2, \tilde{H}_1, \Psi_2) + a(\Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2, \tilde{H}_1, \Psi_1 - \Psi_2) + a(\Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) + a(\Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&= a(\tilde{H}_2 + \Theta, \tilde{H}_1 - \tilde{H}_2, \Psi_1 - \Psi_2) \\
&\leq |\tilde{H}_2 + \Theta|_\infty |\nabla(\tilde{H}_1 - \tilde{H}_2)|_2 |\nabla(\Psi_1 - \Psi_2)|_2
\end{aligned}$$

and thanks to Lemma 3 we arrive to

$$|\nabla(\tilde{H}_1 - \tilde{H}_2)|_2 \leq \frac{1}{\lambda}|\Theta|_{\infty,\Gamma}|\nabla(\Psi_1 - \Psi_2)|_2. \tag{3.9}$$

Now, the estimates (3.8) and (3.9) give

$$\begin{aligned}
\|\mathbf{F}(\Psi_1, H_1) - \mathbf{F}(\Psi_2, H_2)\|_{\mathbf{W}} &= \kappa|\nabla(\tilde{\Psi}_1 - \tilde{\Psi}_2)|_2 + |\nabla \tilde{H}_1 - \tilde{H}_2|_2 \\
&\leq \kappa C|K|_\infty |\nabla(H_1 - H_2)|_2 + \tfrac{1}{\lambda}|\Theta|_{\infty,\Gamma}|\nabla(\Psi_1 - \Psi_2)|_2 \\
&\leq \max\left\{\kappa C|K|_\infty \ ; \ \tfrac{1}{\kappa\lambda}|\Theta|_{\infty,\Gamma}\right\}\|(\Psi_1, H_1) - (\Psi_2, H_2)\|_{\mathbf{W}}.
\end{aligned}$$

In order to have the best constant, we choose $\kappa = \sqrt{\frac{|\Theta|_{\infty,\Gamma}}{\lambda C|K|_\infty}}$ and we obtain

$$\|\mathbf{F}(\Psi_1, H_1) - \mathbf{F}(\Psi_2, H_2)\|_{\mathbf{W}} \leq \sqrt{\tfrac{C|K|_\infty|\Theta|_{\infty,\Gamma}}{\lambda}}\,\|(\Psi_1, H_1) - (\Psi_2, H_2)\|_{\mathbf{W}}.$$

It follows that $\mathbf{F} : \mathbf{D} \to \mathbf{W}$ is Lipschitz continuous, with the Lipschitz constant

$$\beta = \sqrt{\tfrac{C|K|_\infty|\Theta|_{\infty,\Gamma}}{\lambda}}.$$

Since $\mathbf{D}$ is dense in the Banach space $\mathbf{W}$, the map $\mathbf{F}$ can be extended to $\overline{\mathbf{F}} : \mathbf{W} \to \mathbf{W}$ which is still Lipschitz continuous, with the same Lipschitz constant β.

If now $\beta < 1$, then $\overline{\mathbf{F}}$ is a *contraction* and hence has a unique fixed point, say (Ψ_*, H_*). Let us show that (Ψ_*, H_*) is then the unique weak solution of the problem (1.1)-(1.4). By density, there exists a sequence $(\Psi_n, H_n) \in \mathbf{D}$ such that $(\Psi_n, H_n) \to (\Psi_*, H_*)$ in $\mathbf{W}$. In other words, we have

$$\Psi_n \to \Psi_* \quad \text{and} \quad H_n \to H_* \ \text{ in } \ H^1(\Omega) \ \text{ as } \ n \to +\infty. \tag{3.10}$$

If we set $(\tilde{\Psi}_n, \tilde{H}_n) = \mathbf{F}(\Psi_n, H_n)$, then

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla\tilde{\Psi}_n, \nabla u) = -(K.\nabla H_n, u) + (F, u), \tag{3.11}$$

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla\tilde{H}_n, \nabla v) - a(\tilde{H}_n, v, \Psi_n) = -a(v, \Theta, \Psi_n). \tag{3.12}$$

Since $(\tilde{\Psi}_n, \tilde{H}_n) \to \overline{\mathbf{F}}(\Psi_*, H_*) = (\Psi_*, H_*)$ in $\mathbf{W}$, by taking the limits as $n \to +\infty$ in (3.11), we obtain

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla\Psi_*, \nabla u) = -(K.\nabla H_*, u) + (F, u). \tag{3.13}$$

As $n \to +\infty$, we also have $(\nabla \tilde{H}_n, \nabla v) \to (\nabla H_*, \nabla v)$ and $a(v, \Theta, \Psi_n) \to a(v, \Theta, \Psi_*)$, for all $v \in H_0^1(\Omega)$. It follows that $a(\tilde{H}_n, v, \Psi_n)$ has a finite limit as $n \to +\infty$. To compute this limit, let us extract from $(\tilde{H}_n)$ a subsequence $(\tilde{H}_{n_k})$ such that

$$\tilde{H}_{n_k}(x) \to H_*(x) \text{ a.e. in } \Omega \text{ as } k \to +\infty.$$

From Lemma 3, the sequence $(\tilde{H}_n)$ is bounded in $L^\infty(\Omega)$ by some constant $c = c(\Theta)$, and thus $H_* \in L^\infty(\Omega)$ and we have $|H_*|_\infty \le c$. Therefore, on one hand, from (3.10), we have

$$|a(\tilde{H}_n, v, \Psi_n - \Psi_*)| \le c\|v\|\|\Psi_n - \Psi_*\| \to 0 \text{ as } n \to +\infty,$$

and, on the other hand, from the Lebesgue theorem, it holds

$$a(\tilde{H}_{n_k}, v, \Psi_*) \to a(\tilde{H}_*, v, \Psi_*) \text{ as } k \to +\infty,$$

It follows that

$$a(\tilde{H}_n, v, \Psi_n) \to a(\tilde{H}_*, v, \Psi_*) \text{ as } n \to +\infty,$$

and (3.12) gives

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla H_*, \nabla v) - a(H_*, v, \Psi_*) = -a(v, \Theta, \Psi_*). \tag{3.14}$$

From (3.13), (3.14) and the fact that $H_* \in L^\infty(\Omega)$, we obtain that (Ψ_*, H_*) is a weak solution of the problem (1.1)-(1.4).

The uniqueness follows from the fact that any weak solution of (1.1)-(1.4) is a fixed point of $\overline{\mathbf{F}}$. In fact, let (Ψ, H) be a weak solution of the problem (1.1)-(1.4), and $(\Psi_n, H_n) \in \mathbf{D}$ be a sequence such that

$$\Psi_n \to \Psi \text{ and } H_n \to H \text{ in } H^1(\Omega) \text{ as } n \to +\infty.$$

Let us set $(\tilde{\Psi}, \tilde{H}) = \overline{\mathbf{F}}(\Psi, H)$ and $(\tilde{\Psi}_n, \tilde{H}_n) = \mathbf{F}(\Psi_n, H_n)$. Arguing as above, we can take the limits as $n \to +\infty$ in (3.11)-(3.12), and we obtain

$$\forall u \in H_0^1(\Omega, \Gamma_1), \quad (\nabla \tilde{\Psi}, \nabla u) = -(K.\nabla H, u) + (F, u),$$

$$\forall v \in H_0^1(\Omega), \quad \lambda(\nabla \tilde{H}, \nabla v) - a(\tilde{H}, v, \Psi) = -a(v, \Theta, \Psi).$$

This immediatly gives that $\tilde{\Psi} = \Psi$ and $\tilde{H} = H$.

To summarize, we have proved the following result.

Theorem 1. *Let C be the Poincaré constant of Ω. If we have*

$$C\,|K|_\infty\,|\Theta|_{\infty,\Gamma} < \lambda, \tag{3.15}$$

then problem (1.1)-(1.4) *has one and only one weak solution.*

Remark 1. The existence of the solution is still ensured even if the contraction is lost. That is to say if $divK \in L^\infty(\Omega)$ and (3.15) is relaxed as

$$C\,|K|_\infty\,|\Theta|_{\infty,\Gamma} \le \lambda, \tag{3.16}$$

the problem (1.1)-(1.4) still has at least one weak solution. Indeed, choose a sequence of real numbers $\lambda_n > \lambda$ which ensure (3.15) if λ_n is replaced by λ. Of course thanks to the above theorem there exist $\Psi_n \in H_0^1(\Omega, \Gamma_1)$ and $H_n \in L^\infty(\Omega) \cap H_0^1(\Omega)$ solutions to problem (1.1)-(1.4) when λ_n is replaced by λ i.e.

$$\begin{aligned} (\nabla\Psi_n, \nabla u) + (K.\nabla H_n, u) &= (F, u), \\ \lambda_n (\nabla H_n, \nabla v) - a(H_n, v, \Psi_n) + a(v, \Theta, \Psi_n) &= 0, \end{aligned}$$

for any $u \in H_0^1(\Omega, \Gamma_1)$ and any $v \in H_0^1(\Omega)$. Arguing as in [4, Lemma 3.2], we may show that the sequences Ψ_n and H_n are bounded in $H_0^1(\Omega, \Gamma_1)$ and $L^\infty(\Omega) \cap H_0^1(\Omega)$ respectively. Then there exist $\Psi \in H_0^1(\Omega, \Gamma_1)$ and $H \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that -up to a subsequence- we have

$$\begin{aligned} \Psi_n &\rightharpoonup \Psi,\ H_n \rightharpoonup H \ \text{ in } H^1(\Omega), \\ \Psi_n &\to \Psi,\ H_n \to H \ \text{ in } L^2(\Omega), \end{aligned}$$

as $n \to +\infty$. These allow us to pass to the limit in the above system when $n \to +\infty$ and end up with an existence result if the hypothesis (3.15) is relaxed to (3.16).

Acknowledgement. The authors would like to thank Professor Michel Chipot for his comments and helpful suggestions.

References

[1] S. Akesbi, B. Brighi and J.-D. Hoernel : *Steady free convection in a bounded and saturated porous medium.* Proceedings of the 2004 Swiss-Japanese Seminar "Recent advances on Elliptic and Parabolic Issues" World Scientific Publishing Co. Pte. Ltd. (2006) 1–17.

[2] B. Brighi, *The equation* $f''' + ff'' + \mathbf{g}(f') = 0$ *and the associated boundary value problems.* Results. Math. **61** (2012), no. 3-4, 355–391.

[3] B. Brighi, A. Fruchard and T. Sari, *On the Blasius problem*, Adv. Differential Equations, **13** (2008), no. 5-6, 509–600.

[4] B. Brighi and S. Guesmia, *Existence of Solutions and Iterative Approximations for Nonlinear Systems Arising in Free Convection.* Anal. Appl. **7** (2009) no. 3, 225–241.

[5] M. A. Chaudhary, J.H. Merkin and I. Pop, *Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: I prescribed surface temperature*, European J. Mech. B Fluids **14** (1995), no. 2, 217–237.

[6] M. A. Chaudhary, J.H. Merkin and I. Pop, Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: II prescribed surface heat flux, Heat and Mass Transfer **30** (1995), 341–347.

[7] P. Cheng and W. J. Minkowycz, *Free-convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike*, J. Geophys. Res. **82** (1977), no. 14, 2040–2044.

[8] E. I. Ene and D. Poliševski, *Thermal flow in porous media*, D. Reidel Publishing Company, Dordrecht, 1987.

[9] J. H. Merkin and G. Zhang, *On the similarity solutions for free convection in a saturated porous medium adjacent to impermeable horizontal surfaces*, Wärme und Stoffübertr. **25** (1990), 179–184.

[10] R. A. Wooding, *Convection in a saturated porous medium at large Rayleigh number or Peclet number*, J. Fluid. Mech. **15** (1963), 527–544.

Laboratoire de Mathématiques, Informatique et Applications, Université de Haute Alsace, 4, rue des Frères Lumière, 68093 Mulhouse, France
E-mail address: bernard.brighi@uha.fr

Mathematics department, College of sciences, Qassim University, Elqassim, Saudi Arabia
E-mail address: guesmia@math.uzh.ch

JEPE Vol 1, 2015, p. 271-279

Orthogonal $\perp$

ON THE EXPONENTIAL CONVERGENCE RATE FOR A NON-GRADIENT FOKKER-PLANCK EQUATION IN COMPUTATIONAL NEUROSCIENCE

J. A. CARRILLO, S. MANCINI, AND M.-B. TRAN

ABSTRACT. This paper concerns the proof of the exponential rate of convergence of the solution of a Fokker-Planck equation, with a drift term not being the gradient of a potential function and endowed by Robin type boundary conditions. This kind of problem arises, for example, in the study of interacting neurons populations. Previous studies have numerically shown that, after a small period of time, the solution of the evolution problem exponentially converges to the stable state of the equation.

1. INTRODUCTION

The exponential convergence of the solutions to evolution problems towards the steady states has been largely addressed for many years, see for example [1, 5, 7, 9, 10, 11, 12]. Techniques and proofs are usually based on the nature of each equation. However, the general entropy method developed in [12] for linear problems and used in computational neuroscience in [6, 8] is a powerful method for investigation of this question in the problem under consideration here.

In this work, we are concerned with the mathematical proof of the exponential rate of convergence of the solution of a non-gradient Fokker-Planck equation towards its stationary state. This problem arises, for instance, in the modeling of the evolution of the firing rates of two populations of interacting neurons. In this framework, the partial differential equation describing the evolution of the probability distribution function $p = p(t, \nu)$, with $\nu = (\nu_1, \nu_2) \in \Omega$ and $t \geq 0$ is the Fokker-Planck equation (or forward Kolmogorov equation):

$$\partial_t p + \nabla \cdot \left(Fp - \frac{\beta^2}{2} \nabla p \right) = 0\,, \quad \text{in } [0, \infty[\times \Omega$$

where Ω is a bounded domain of $\mathbb{R}^2$ and the vector field $F = F(\nu)$ is defined by:

$$F = \begin{pmatrix} -\nu_1 + \phi(\lambda_1 + w_{11}\nu_1 + w_{12}\nu_2) \\ -\nu_2 + \phi(\lambda_2 + w_{21}\nu_1 + w_{22}\nu_2) \end{pmatrix}$$

2010 *Mathematics Subject Classification.* Primary: 35Q84; Secondary: 35K99.

Key words and phrases. non-gradient Fokker-Planck equation, Exponential convergence, Poincaré coefficient.

Received 23/03/2016, accepted 25/03/2016.

JAC was partially supported by the Royal Society by a Wolfson Research Merit Award and the EPSRC grant EP/K008404/1. SM was partially supported by the KIBORD project (ANR-13-BS01-0004) funded by the French Ministry of Research. MBT was partially supported by the NSF Grant RNMS (Ki-Net) 1107444 and the ERC Advanced Grant FP7-246775 NUMERIWAVES..

with $\phi(z)$ a sigmoid function and w_{ij} positive weights. The Fokker-Planck equation is endowed by the following Robin or no flux boundary conditions:

$$\left(Fp - \frac{\beta^2}{2}\nabla p\right) \cdot n = 0\,, \text{ on } \partial\Omega$$

with n the outward normal unit vector on $\partial\Omega$, β being the noise level, and a non-negative initial data $p(0,\nu) = p_0(\nu) \geq 0$. It is easily seen that the F does not satisfy the Schwarz conditions, that is $\partial_{\nu_2} F_1 \neq \partial_{\nu_1} F_2$, so that there exists no potential function $V(\nu)$ such that $F = -\nabla V$. This in particular implies that it is less likely that there is an explicit formulation of the associated steady state, as it would be the case with a drift F defined by $F = -\nabla V$ for some smooth potential function V. We refer to [2, 3, 4] for some discussion on these issues and some cases in which, despite of the non-gradient drifts in the Fokker-Planck equations, one is able to find explicit stationary states. In our case, for general domains and so complicated drift it is in general not possible to find explicitly these stationary states. Note that as soon as we find a potential function V such that the drift $F = -\nabla V$, the steady state reads

$$p_\infty = C \exp\left(\frac{-2V}{\beta^2}\right),$$

with C a normalization coefficient.

In this work, we will deal with estimates of the exponential convergence rates for general linear Fokker-Planck equations with non-gradients drifts of the form:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (Fu) = 0 \quad \text{ in } \Omega \times (0,T), \\ u(0,\cdot) = u_0(\cdot) \quad \text{ in } \Omega, \\ (Fu - \nabla u) \cdot n = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

where the unknown $u(t,x)$ is a probability density function, the drift $F \in C^2(\overline{\Omega})$ satisfies the incoming boundary condition

$$F \cdot n < 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

and the initial data is normalized to 1, $\int_\Omega u_0(x)\,dx = 1$. We have assumed unit diffusion constant for simplicity without loss of generality. Under these hypotheses, existence, uniqueness and positivity of the solution $u = u(t,x)$ of the evolution problem (1.1), and of the stable state (stationary solution) $u_\infty(x)$ of the associated problem were proved in [8, Theorem 2], as well as the mass density conservation,

$$\int_\Omega u(t,x)\;dx = \int_\Omega u_0(x)\;dx = 1,$$

and the L^2-convergence of the time dependent solution to the stationary solution u_∞. Numerical simulations were also performed underlying the exponential rate of convergence of the solution of the evolution problem towards the stable state. However, the theoretical proof of this exponential rate of convergence was not discussed in [8]. In the next section we will discuss the exponential convergence towards the steady state u_∞ by direct Poincaré's inequalities. Section 3 is devoted to an alternative approach implying relations between the sharp exponential convergence rates with the best constants for different Poincaré's type inequalities.

2. Convergence to equilibrium

Consider a drift $F \in C^2(\overline{\Omega})$ such that $F \cdot n < 0$ and let $u = u(t,x)$ be the unique solution of problem (1.1), and $u_\infty = u_\infty(x)$ be the solution of the stationary associated problem:

$$\begin{cases} -\Delta u_\infty + \nabla(F u_\infty) = 0 & \text{in } \Omega, \\ (F u_\infty - \nabla u_\infty) \cdot n = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.1}$$

ensured by [8, Theorem 2] and satisfying

$$\int_\Omega u_0(x)\,dx = \int_\Omega u_\infty(x)\,dx = 1.$$

We shall prove the following result:

Theorem 1. *The solution u to* (1.1) *exponentially converges to the steady state u_∞: there exist α, $C > 0$ such that*

$$\|u - u_\infty\|_{L^2(\Omega)} \leq C \exp(-\alpha t).$$

Let us first remind the reader that applying the General Entropy Method [12] adapted to this problem in [8, Theorem 4] one obtains

$$\frac{d}{dt}\mathcal{H}(u|u_\infty) = -\mathcal{D}(u|u_\infty) \leq 0, \tag{2.2}$$

where

$$\mathcal{H}(u|u_\infty) = \int_\Omega \frac{(u-u_\infty)^2}{u_\infty}\,dx$$

and

$$\mathcal{D}(u|u_\infty) = \int_\Omega u_\infty \left|\nabla \frac{u}{u_\infty}\right|^2 dx\,.$$

Therefore, in order to get the exponential convergence, the key point to prove is the following Poincaré inequality:

$$\mathcal{H}(u|u_\infty) = \int_\Omega \frac{(u-u_\infty)^2}{u_\infty}\,dx \leq \mathcal{P}_\Omega(u_\infty) \int_\Omega u_\infty \left|\nabla \frac{u}{u_\infty}\right|^2 dx = \mathcal{P}_\Omega(u_\infty)\,\mathcal{D}(u|u_\infty), \tag{2.3}$$

where $\mathcal{P}_\Omega(u_\infty) > 0$ is the best constant for the Poincaré inequality (2.3), with weight u_∞ in the domain Ω. In the sequel, we will not put the subindex Ω in the Poincaré constants for notational simplicity. In fact, from (2.2) and (2.3) we then deduce that

$$\frac{d}{dt}\mathcal{H}(u|u_\infty) + \mathcal{P}(u_\infty)\ \mathcal{H}(u|u_\infty) \leq 0,$$

and applying Gronwall's Lemma

$$\mathcal{H}(u|u_\infty) \leq C \exp(-\alpha t),$$

with $C = \mathcal{H}(u_0|u_\infty)$ and $\alpha = \mathcal{P}(u_\infty)$, leading to the exponential convergence of u to u_∞. Notice that (2.3) is equivalent to

$$\int_\Omega u_\infty \left(\frac{u}{u_\infty} - 1\right)^2 dx \leq \mathcal{P}(u_\infty) \int_\Omega u_\infty \left|\nabla \left(\frac{u}{u_\infty} - 1\right)\right|^2 dx\,, \tag{2.4}$$

Put $w = \frac{u}{u_\infty} - 1$, then (2.4) reads

$$\int_\Omega u_\infty w^2\, dx \le \mathcal{P}(u_\infty) \int_\Omega u_\infty \left|\nabla w\right|^2\, dx\,.$$

Note that, since both u and u_∞ are normalized to 1, we also have the constraint:

$$\int_\Omega u_\infty w\, dx = \int_\Omega u_\infty \left(\frac{u}{u_\infty} - 1\right)\, dx = \int_\Omega (u - u_\infty)\, dx = 0.$$

Inequality (2.3) then follows from the following result:

Proposition 1. *Under the assumption*

$$\int_\Omega \Phi H\, dx = 0,$$

where H is bounded from above and below by positive constants, there exists a sharp constant $\mathcal{P}(H) > 0$ such that the following Poincaré inequality holds

$$\int_\Omega H|\Phi|^2\, dx \le \mathcal{P}(H) \int_\Omega H|\nabla\Phi|^2\, dx.$$

Proof: We follow the classical proof of Poincaré's inequality, by contradiction. Suppose that there exists a sequence $\{\Phi_m\}$, such that

$$\max(H)\int_\Omega |\Phi_m|^2\, dx \ge \int_\Omega H|\Phi_m|^2\, dx \ge m \int_\Omega H|\nabla\Phi_m|^2\, dx \ge m \min(H)\int_\Omega |\nabla\Phi_m|^2\, dx.$$

By normalization, we can suppose that $\|\Phi_m\|_{L^2} = 1$. Therefore the sequence $\{\Phi_m\}$ is bounded in $W^{1,2}(\Omega)$. By mean of the Rellich-Kondrachov Theorem, there exists a subsequence $\{\Phi_{m_j}\}$ and a function $\bar\Phi$ in $L^2(\Omega)$ such that $\{\Phi_{m_j}\}$ converges strongly to $\bar\Phi$ in $L^2(\Omega)$. Passing to the limit $m \to \infty$, we get that $\nabla\bar\Phi = 0$ a.e. and $\|\bar\Phi\|_{L^2} = 1$, which implies $\bar\Phi$ is a constant. But, since H is positive, this contradicts the hypothesis $\int_\Omega \bar\Phi H\, dx = 0$, concluding the proof. ∎

Although we can obviously relate the best constant $\mathcal{P}(u_\infty)$ to the classical Poincaré inequality best constant $\mathcal{P}(1)$ by $\mathcal{P}(u_\infty) \le \frac{\max(u_\infty)}{\min(u_\infty)}\mathcal{P}(1)$, we have no information on the value of the constant obtained by this proof. In the next section, we propose an alternative proof in which we show that the constants C and α in Theorem 1 are linked to the bounds of the solution K of an auxiliary problem.

3. Alternative Proof

In this section, we propose an alternative proof of Theorem 1 which gives us another characterization of the constants C and α. Note first that to prove the exponential convergence of u to u_∞, is equivalent to prove the exponential convergence of $\Phi = u/u_\infty$ to 1. As we will see, the proof is based on a new conservation property, see Proposition 3 below.

Before dealing with the exponential convergence problem for the function Φ, we need to define for which problem Φ is a solution. This is done in the following. Let us first define Φ and the initial data Φ_0 as follows, since $u_\infty > 0$, we define

$$\Phi(t,x) = \frac{u(t,x)}{u_\infty(x)}, \quad \Phi_0(x) = \frac{u_0(x)}{u_\infty(x)}\,,$$

and let us introduce a modified drift F^* as

$$F^* = F - 2\frac{\nabla u_\infty}{u_\infty}. \tag{3.1}$$

We note that on the boundary $\partial\Omega$, we obtain

$$F^* \cdot n = \left(F - 2\frac{\nabla u_\infty}{u_\infty}\right) \cdot n = -F \cdot n > 0,$$

since $F \cdot n = (\nabla u_\infty / u_\infty) \cdot n$ and F satisfies (1.2).

Proposition 2. *The function $\Phi = u/u_\infty$ satisfies*

$$\begin{cases} \partial_t \Phi - \Delta\Phi + F^* \cdot \nabla\Phi = 0 \ \textit{in}\ \Omega \times (0,T), \\ \Phi(.,0) = \Phi_0(.) \ \textit{in}\ \Omega, \\ \nabla\Phi \cdot n = 0 \ \textit{on}\ \partial\Omega. \end{cases} \tag{3.2}$$

Proof: In order to simplify the notation, consider $g = 1/u_\infty$, $g \in C^2(\Omega) \cap C(\bar{\Omega})$ and $g > 0$, and $\Phi = u\ g$ and $\Phi_0 = u_0\ g$. We derive $\Phi = u\ g$ to get, for all $i = 1, 2$

$$\partial_{x_i}\Phi = g\partial_{x_i}u + u\partial_{x_i}g,$$

$$\partial^2_{x_i}\Phi = \partial^2_{x_i}ug + \Phi\frac{\partial^2_{x_i}g}{g} + 2\frac{\partial_{x_i}g}{g}\partial_{x_i}\Phi - 2\Phi\left|\frac{\partial_{x_i}g}{g}\right|^2.$$

These identities imply, for the boundary conditions on $\partial\Omega$

$$\nabla\Phi \cdot n = \left(\frac{\nabla u}{u_\infty} - \frac{u\nabla u_\infty}{u^2_\infty}\right) \cdot n = \left(\frac{\nabla u}{u_\infty} - \frac{uF}{u_\infty}\right) \cdot n = 0,$$

and inside the domain Ω, we obtain

$$\begin{aligned} \partial_t\Phi - \Delta\Phi &= (\partial_t u - \Delta u)g - \Phi\frac{\Delta g}{g} - 2\nabla\Phi \cdot \frac{\nabla g}{g} + 2\Phi\left|\frac{\nabla g}{g}\right|^2 \\ &= -\nabla \cdot (Fu)g - \Phi\frac{\Delta g}{g} - 2\nabla\Phi \cdot \frac{\nabla g}{g} + 2\Phi\left|\frac{\nabla g}{g}\right|^2 \\ &= -\nabla \cdot (F\Phi) - 2\nabla\Phi \cdot \frac{\nabla g}{g} \\ &\quad + \Phi\left[-\left(\frac{\Delta g}{g} - 2\left|\frac{\nabla g}{g}\right|^2\right) + F \cdot \frac{\nabla g}{g}\right]. \end{aligned} \tag{3.3}$$

Since $u_\infty = 1/g$, we have

$$\nabla u_\infty = -\frac{\nabla g}{g^2} \quad \text{and} \quad \Delta u_\infty = -\frac{\Delta g}{g^2} + 2\frac{|\nabla g|^2}{g^3},$$

so that, dividing (3.3) by $g = 1/u_\infty$, we get

$$\begin{aligned} u_\infty\partial_t\Phi - u_\infty\Delta\Phi &+ u_\infty F \cdot \nabla\Phi - 2\nabla\Phi \cdot \nabla u_\infty \\ &+ \Phi[-\Delta u_\infty + F \cdot \nabla u_\infty + u_\infty \nabla \cdot F] = 0. \end{aligned}$$

Hence Φ must satisfy

$$
\begin{cases}
u_\infty \partial_t \Phi - u_\infty \Delta\Phi + u_\infty F \cdot \nabla\Phi - 2\nabla\Phi \cdot \nabla u_\infty \\
\qquad\qquad +\Phi[-\Delta u_\infty + \nabla\cdot(F u_\infty)] = 0 \quad \text{in} \;\; \Omega\times(0,T), \\
\Phi(\cdot,0) = \Phi_0(\cdot) \quad \text{in} \;\; \Omega, \\
\nabla\Phi\cdot n = 0 \quad \text{on} \;\; \partial\Omega.
\end{cases}
\tag{3.4}
$$

Finally, recalling the definition of F^* given by (3.1), and that u_∞ solves (2.1), problem (3.4) is converted into (3.2), concluding the proof. ∎

Consider now the stationary problem associated to (3.2):

$$
\begin{cases}
-\Delta\Phi_\infty + F^*\cdot\nabla\Phi_\infty = 0 \quad \text{in} \;\; \Omega\times(0,T), \\
\nabla\Phi_\infty\cdot n = 0 \quad \text{on} \;\; \partial\Omega.
\end{cases}
\tag{3.5}
$$

then it is easily seen that the constants are solutions to (3.5), so that proving the exponential convergence of Φ to $\Phi_\infty = 1$, will give us the wanted exponential convergence of u to u_∞. So, the main result we have to prove, yielding to the exponential convergence of u to u_∞ is the following:

Theorem 2. *The solution Φ of* (3.2) *exponentially converges in time to its equilibrium state Φ_∞, solution to* (3.5).

In the sequel, in order to prove Theorem 2, we need an auxiliary problem, which is the dual problem to (3.5) given by

$$
\begin{cases}
\Delta K + \nabla\cdot(KF^*) = 0 \quad \text{in} \;\; \Omega, \\
(\nabla K + KF^*)\cdot n = 0 \quad \text{on} \;\; \partial\Omega,
\end{cases}
\tag{3.6}
$$

for which we next prove a conservation property of the function ΦK. This conservation results is a key point in order to prove a Poincaré's inequality on the function Φ and to get in an alternative way the exponential convergence.

Proposition 3. *Let K be a solution of* (3.6) *and Φ be a solution of* (3.2), *then*

$$\int_\Omega \Phi_0 K\,dx = \int_\Omega \Phi K\,dx.$$

Proof: Note that $-F^*\cdot n < 0$ on $\partial\Omega$, thus by using the same arguments as in [8], this problem admits a unique positive and bounded solution $K \in H^2(\Omega)$. Furthermore, we have

$$-\Delta\Phi + F^*\nabla\Phi = -\frac{1}{K}\nabla\cdot(K\nabla\Phi) + \left(\frac{\nabla K}{K} + F^*\right)\nabla\Phi,$$

so that problem (3.2) can be written as

$$
\begin{cases}
\partial_t\Phi - \dfrac{1}{K}\nabla\cdot(K\nabla\Phi) + \left(\dfrac{\nabla K}{K} + F^*\right)\cdot\nabla\Phi = 0 \quad \text{in} \;\; \Omega\times(0,T), \\
\Phi(.,0) = \Phi_0(.) \quad \text{in} \;\; \Omega, \\
\nabla\Phi\cdot n = 0 \quad \text{on} \;\; \partial\Omega.
\end{cases}
\tag{3.7}
$$

Use K as a test function for (3.7), then

$$\begin{aligned}0 &= \partial_t \int_\Omega \Phi K\, dx + \int_\Omega (\nabla K + KF^*) \cdot \nabla \Phi\, dx - \int_{\partial\Omega} K \nabla \Phi \cdot n\, d\sigma(x)\\ &= \partial_t \int_\Omega \Phi K\, dx - \int_\Omega \nabla \cdot (\nabla K + KF^*)\Phi\, dx + \int_{\partial\Omega} \Phi(\nabla K + KF^*) \cdot n\, d\sigma(x)\\ &= \partial_t \int_\Omega \Phi K\, dx - \int_\Omega (\Delta K + \nabla \cdot (KF^*))\Phi\, dx = \partial_t \int_\Omega \Phi K\, dx,\end{aligned}$$

which concludes the proof. ∎

Since, problem (3.6) admits a unique solution, then the constants are the unique solutions, in distributional sense, of the stationary problem (3.5). Hence, up to a normalization constant, we shall consider in the sequel that $\Phi_\infty = 1$. Moreover, if we consider $\Psi = \Phi - \Phi_\infty$, then Ψ still satisfies (3.2) and converges to 0, in other words its equilibrium is the null function. Therefore, renaming $\Psi = \Phi$ and in order to simplify the proof of Theorem 2, we are reduced to prove the exponential convergence of Φ solution of (3.2) to 0, which is done in Theorem 3 below. We impose the following normalization

$$\int_\Omega \Phi_0 K\, dx = 0,$$

so that from the conservation property in Proposition 3 we get

$$\int_\Omega \Phi K\, dx = 0.$$

Thanks to this normalization, the proof of Theorem 2 is reduced to prove the following result.

Theorem 3. *Assuming that*

$$\int_\Omega \Phi_0 K\, dx = 0,$$

then, the solution Φ *of* (3.2) *exponentially decays in time towards* 0*, that is,*

$$\|\Phi\|_{L^2(\Omega)} \leq \tilde{C} \exp(-\tilde{\alpha} t),$$

where $\tilde{C}$ *and* $\tilde{\alpha}$ *are estimated by*

$$\tilde{C} = \frac{1}{\min(K)} \int_\Omega |\Phi_0|^2 K\, dx \quad \text{and} \quad \tilde{\alpha} = \mathcal{P}(K)\,.$$

Proof: Use $K\Phi$ as a test function for (3.7). Since K is bounded from above and from below by some positive constants, and since from the normalization hypothesis, the Poincaré inequality of Proposition 1 holds, then:

$$
\begin{aligned}
0 &= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\,dx + \int_\Omega |\nabla\Phi|^2 K\,dx + \int_\Omega \Phi(\nabla K + KF^*)\cdot\nabla\Phi\,dx \\
&= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\,dx + \int_\Omega |\nabla\Phi|^2 K\,dx + \frac{1}{2}\int_\Omega (\nabla K + KF^*)\cdot\nabla|\Phi|^2\,dx \\
&= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\,dx + \int_\Omega |\nabla\Phi|^2 K\,dx - \frac{1}{2}\int_\Omega (\Delta K + \nabla\cdot(KF^*))|\Phi|^2\,dx \\
&= \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\,dx + \int_\Omega |\nabla\Phi|^2 K\,dx \\
&\geq \frac{1}{2}\partial_t \int_\Omega |\Phi|^2 K\,dx + \mathcal{P}(K)\int_\Omega |\Phi|^2 K\,dx\,. \qquad (3.8)
\end{aligned}
$$

Inequality (3.8) implies:

$$
\int_\Omega |\Phi|^2 K\,dx \leq \left(\int_\Omega |\Phi_0|^2 K\,dx\right)\exp(-2\tilde{\alpha}t),
$$

with $\tilde{\alpha} = \mathcal{P}(K)$. With some more computations, we finally obtain

$$
\int_\Omega |\Phi|^2\,dx \leq \frac{1}{\min(K)}\int_\Omega |\Phi|^2 K\,dx \leq \left(\frac{1}{\min(K)}\int_\Omega |\Phi_0|^2 K\,dx\right)\exp(-2\tilde{\alpha}t),
$$

concluding the proof of theorem 3. ∎

Finally, to conclude the proof of the time exponential convergence of u, solution to (1.1), to the steady state u_∞ given by (2.1), we just have to note that, since $g = 1/u_\infty$ is strictly positive and bounded, Theorem 2 leads to:

$$
\|u - u_\infty\|_{L^2(\Omega)} \leq \max(u_\infty)\tilde{C}\exp(-\tilde{\alpha}t)\,,
$$

concluding the proof of Theorem 1. Notice that the decay constants might be different to the ones in Theorem 1 obtained in Section 2 due to the different stationary problems with solutions K and u_∞ used.

References

[1] N. D. Alikakos and P. W. Bates. Stabilization of solutions for a class of degenerate equations in divergence form in one space dimension. *J. Differential Equations*, 73(2):363–393, 1988.

[2] A. Arnold, J. A. Carrillo, and C. Manzini. Refined long-time asymptotics for some polymeric fluid flow models. *Commun. Math. Sci.*, 8(3):763–782, 2010.

[3] A. Arnold and E. Carlen. A generalized Bakry-Emery condition for non-symmetric diffusions. In *International Conference on Differential Equations, Vol. 1, 2 (Berlin, 1999)*, pages 732–734. World Sci. Publ., River Edge, NJ, 2000.

[4] A. Arnold, E. Carlen, and Q. Ju. Large-time behavior of non-symmetric Fokker-Planck type equations. *Commun. Stoch. Anal.*, 2(1):153–175, 2008.

[5] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter. On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100, 2001.

[6] M. J. Cáceres, J. A. Carrillo, and B. Perthame. Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states. *J. Math. Neurosci.*, 1:Art. 7, 33, 2011.

[7] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani, and A. Unterreiter. Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities. *Monatsh. Math.*, 133(1):1–82, 2001.

[8] J. A. Carrillo, S. Cordier, and S. Mancini. A decision-making Fokker-Planck model in computational neuroscience. *J. Math. Biol.*, 63(5):801–830, 2011.

[9] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159(2):245–316, 2005.

[10] A. Haraux. Comportement à l'infini pour une équation d'ondes non linéaire dissipative. *C. R. Acad. Sci. Paris Sér. A-B*, 287(7):A507–A509, 1978.

[11] M. W. Hirsch. Differential equations and convergence almost everywhere in strongly monotone semiflows. In *Nonlinear partial differential equations (Durham, N.H., 1982)*, volume 17 of *Contemp. Math.*, pages 267–285. Amer. Math. Soc., Providence, R.I., 1983.

[12] P. Michel, S. Mischler, and B. Perthame. General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appl. (9)*, 84(9):1235–1260, 2005.

Department of Mathematics, Imperial College London, London SW7 2AZ, UK

MAPMO, UMR 7349, CNRS, Université of Orléans, Route de Chartres, BP. 6759, 45067 Orléans, France

Department of Mathematics, University of Wisconsin Madison, WI 53706, USA

JEPE Vol 1, 2015, p. 281-305

Orthogonal $\perp$

TRAVELING SPOTS ON MULTI-DIMENSIONAL EXCITABLE MEDIA

YAN-YU CHEN, HIROKAZU NINOMIYA, AND RYOTARO TAGUCHI

Abstract. In this paper, the existence of a traveling spot on multi-dimensional excitable media is studied. First, the equations for the front and the back of a traveling spot including the interface equation are derived from the singular limit analysis of the reaction-diffusion system of the FitzHugh-Nagumo type. Then, the existence and uniqueness for the front and the back are shown when the speed of the traveling spot is less than the one of a planar wave. In addition, the non-convexity of the traveling spots is shown depending on the speed. Finally, the traveling spot converges to the planar wave as its speed tends to the one of the planar wave, which means that a traveling spot connects a stationary solution and a planar wave.

1. Introduction

Self-organized pattern formation has attracted considerable attention in recent years. In particular, various spatio-temporal patterns in excitable media have been reported in the last three decades, e.g., [12, 17, 20, 21]. Excitable media possess two states: exciting state and resting state. Localized patterns are produced by the exciting states. To study localized patterns such as a planar wave, a spiral wave, and a spot, several methods have been introduced. Two major methods are bifurcation theory [7, 15, 16] and singular limit analysis [14, 27]. These studies reveal that there is a radially symmetric stationary solution and the asymmetric traveling spot bifurcates from it. Local analysis has been performed theoretically, but the global branches were studied numerically. Namely, the global bifurcation of the singular limit problem remains difficult to address mathematically. New methods in multi-dimensional space have been demanded. Before we state the main results, we review more details of the singular limit problem. The singular limit analysis is a powerful tool to capture patterns. Essentially, by taking an appropriate singular limit of the Allen-Cahn equation, the solution converges to a characteristic function of a moving domain. The moving boundary is called *interface*. It is well known that the interface is ruled by a mean curvature equation with a driving force. This method is applicable to the FitzHugh-Nagumo

2010 *Mathematics Subject Classification.* Primary: 06B10; Secondary: 06D05.

Key words and phrases. Asymptotic, parabolic, travelling spots.

Received 28/02/2016, accepted 29/03/2016.

The authors would like to thank the Mathematics Division of NCTS (Taipei Office) for the support of the second author's visit to Taiwan for which this paper was inspired. The second author is partially supported by the Challenging Exploratory Research (No.25610036), Grant-in-Aid for Scientific Research (B) (No. 26287024) from the Japan Society for the Promotion of Science.

system

$$
\left\{ \begin{array}{rcll} u_t & = & \varepsilon \Delta u + \dfrac{1}{\varepsilon}(f_0(u) - \beta v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t & = & d\Delta v + g_0(u,v), & x \in \mathbb{R}^n,\ t > 0, \end{array} \right. \tag{1.1}
$$

where $\varepsilon > 0$, $d > 0$, $f_0(u) = u(u - 1/2)(1 - u)$, $g_0(u,v) = g_1 u - g_2 v + g_3$, and n is an integer greater than two. For the FitzHugh-Nagumo system, the limiting system consists of the interface equation and the equation for the inhibitor v:

$$
\left\{ \begin{array}{rcll} V & = & W(v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t & = & d\Delta v + g_0(h_\pm(v), v), & x \in \mathbb{R}^n,\ t > 0, \end{array} \right. \tag{1.2}
$$

where V is a normal velocity of the interface, W is a function of v determined by the speed of a one-dimensional traveling wave, and $h_-(v), h_+(v)$ are the roots of $f_0(u) = v$ with $h_-(v) \le h_+(v)$. See [14, 27] for details. In some studies [4, 14], the mean curvature κ of the interface with small coefficient was added to (1.2). To derive the mean curvature term, let us consider the other parameter settings as follows:

$$
\left\{ \begin{array}{rcll} u_t & = & \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t & = & d\Delta v + g_0(u,v), & x \in \mathbb{R}^n,\ t > 0, \end{array} \right. \tag{1.3}
$$

where $f_\varepsilon(u) := u(u - 1/2 + \varepsilon\alpha)(1 - u)$. Then, the corresponding limiting system includes a mean curvature in the first equation of (1.2), namely,

$$
\left\{ \begin{array}{rcll} V & = & W(v) - (n-1)\kappa, & x \in \partial\Omega(t)\ t > 0, \\ v_t & = & d\Delta v + g_0(\chi_\Omega, v), & x \in \mathbb{R}^n,\ t > 0, \end{array} \right. \tag{1.4}
$$

where $\chi_{\Omega(t)}$ is a characteristic function of the domain Ω(t) surrounded by the interface $\partial\Omega(t)$, i.e.,

$$
\chi_{\Omega(t)} = \left\{ \begin{array}{ll} 1, & x \in \Omega(t), \\ 0, & \text{otherwise.} \end{array} \right.
$$

The convergence of the solutions of (1.3) to those of (1.4) is shown in [1] when $d > 0$. However, it is difficult to investigate the behavior of the solution of this limiting problem even in one spatial dimension. It is partially studied in [11] through the aid of the numerical computation.

When $d = 0$, the equation for v is easier to be handled. One-dimensional traveling pulses have been studied by many researchers, for example, [3, 10, 25] and so on. Karma [13] and Pelcé - Sun [24] studied spiral waves in two-dimensional excitable media (see also Scheel [26] and the references therein). They also provide some numerical results of the existence of rotating spiral waves. This system is often called a *wave front interaction model.* In [28], Zykov and Showalter extended this system to the traveling spot. Along these lines, we consider the multi-dimensional excitable media. However, (1.3) becomes a bistable system because there are three intersection points between the graph $v = f_\varepsilon(u)/(\varepsilon\beta)$ and the nullcline $g_0(u,v) = 0$ for small $\varepsilon > 0$. Therefore (1.4) is not suitable for the excitable

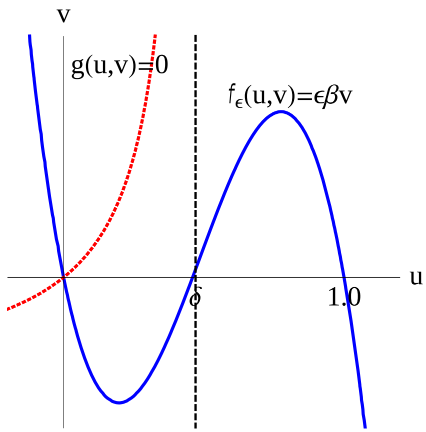


FIGURE 1. Nullclines of nonlinear functions.

media when ε is small. Chen, Kohsaka, and Ninomiya [5] introduced the FitzHugh-Nagumo type reaction-diffusion system with small parameter ε as follows:

$$\tag{1.5} \begin{cases} u_t = \Delta u + \dfrac{1}{\varepsilon^2}(f_\varepsilon(u) - \varepsilon\beta v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t = g(u,v), & x \in \mathbb{R}^n,\ t > 0, \end{cases}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and

$$\tag{1.6} f_\varepsilon(u) := u(1-u)\left(u - \frac{1}{2} + \varepsilon\alpha\right), \quad g(u,v) = g_1 u - \frac{g_2 v}{g_4 + g_3 v}$$

with positive constants $g_1, \cdots, g_4$ and α. The system (1.5) is a modified version of the FitzHugh-Nagumo system (1.3). Under the assumption

$$\frac{g_2}{g_3} < \frac{g_1}{2},$$

the ordinary differential equations corresponding to (1.5) is monostable for any $\varepsilon > 0$. Hence (1.5) is an excitable system for any $\varepsilon > 0$ (see Figure 1). In [5], the following free boundary problem has been formally derived from the singular limit problem of (1.5):

$$\tag{1.7} \begin{cases} V = W(v) - (n-1)\kappa, & x \in \partial\Omega(t), t > 0, \\ v_t = g(\chi_{\Omega(t)}, v), & x \in \mathbb{R}^n, t > 0 \end{cases}$$

where $\Omega(t)$, V, $\chi_{\Omega(t)}$, and κ are defined as above. In the process to derive the singular limit problem, we obtain

$$W(v) = a - bv$$

where $a = \sqrt{2}\alpha$ and $b = 6\sqrt{2}\beta$. This system (1.7) looks similar to (1.4) with $d = 0$. As stated before, even though (1.7) can be formally derived from (1.3), (1.3) becomes bistable for small $\varepsilon > 0$ and (1.4) is anymore realistic as the limit problem of the excitable system.

To look for the traveling spot solution of (1.7), we assume that the exciting region $\Omega(t)$ and the function v do not change their shapes and translate along the x_1-direction with constant speed c. Namely, the traveling spot of (1.7) with speed c satisfies

$$\begin{cases} \Omega(t) = \{(x_1 + ct, x') \in \Omega(0)\}, \\ v(x,t) = v(x_1 - ct, x', 0), \end{cases}$$

after an appropriate translation and rotation where $x' := (x_2, x_3, \cdots, x_n)$. In addition, we assume that $\Omega(t)$ and v are rotationally symmetric with respect to the x_1-axis. Then, we rewrite the free boundary problem (1.7) as

$$\begin{cases} c\cos\theta = W(v) - (n-1)\kappa, & x \in \partial\Omega(0), \\ -cv_{x_1} = g(\chi_{\Omega(0)}, v), & x \in \mathbb{R}^n, \end{cases} \tag{1.8}$$

where θ is the angle of the outer normal vector as measured from the positive x_1-axis, and V is equal to $c\cos\theta$. Therefore, if there exists a solution (Ω, v, c) of (1.8), then $\Omega(t) = \{(x_1 + ct, x') \mid (x_1, x') \in \Omega\}$ and $v(x_1 - ct, x')$ satisfy (1.7).

Theorem 1. *For any $c \in (0, a)$, there exist a unique constant b and a solution (Ω, v, c) of (1.8) such that*

$$\begin{aligned} & v \in C^1(\mathbb{R}^n \setminus \partial\Omega) \cap C^0(\mathbb{R}^n), \qquad \partial\Omega \in C^2, \\ & v(x_1, x') = 0 \quad \text{if } |x'| \geq r_0, \\ & \lim_{R\to\infty} \sup_{|(x_1,x')|\geq R} v(x_1, x') = 0, \end{aligned} \tag{1.9}$$

where r_0 is the distance between the x_1-axis and the boundary $\partial\Omega$ of the rotationally symmetric domain Ω.

Zykov and Showalter [28] studied the two-dimensional traveling spot for (1.4) with $g_0(u, v)$ replaced by $g_0(u, 0)$ and $d = 0$. In this case, we note that $g_0(u, 0)$ vanishes in the resting region. Thus, for their solutions, the concentration of the inhibitor v must be a constant with respect to x_1 after the back. However, for the excitable media such as the photosensitive Belousov-Zhabotinsky experiment, the inhibitor v must be recovered to 0 after the exciting region. Therefore, our singular limit problem (1.7) and the corresponding reaction-diffusion system (1.5) are improved ones.

We also remark that the positivity of the diffusion coefficient is essential in the proof of the local existence of solutions to (1.4) and that the local existence of solutions to (1.7) is still open. Recently, Ninomiya and Wu [22] have shown the local existence of solutions of (1.7) near traveling waves in an appropriate functional space.

The traveling spot obtained in Theorem 1 possesses the following properties.

Theorem 2. *Let (Ω_c, v, c) be a traveling spot in Theorem 1. Then, the following hold:*

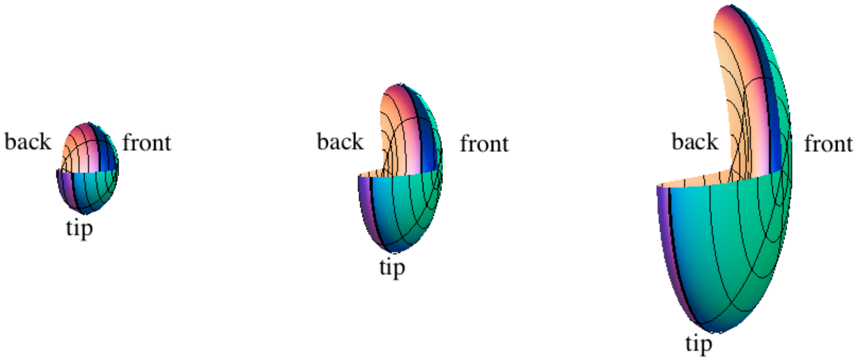
(i) *there is a c_* such that for $c \in [0, c_*)$, Ω_c is convex;*

(ii) *there is a c^* such that for $c \in (c^*, a)$, Ω_c is non-convex;*

(iii) *as $c \searrow +0$, Ω_c converges to the stationary disk $\{x \in \mathbb{R}^n \mid |x| < 1/a\}$;*

(iv) *as $c \nearrow a$, Ω_c converges to the planar wave $\{x = (x_1, x') \in \mathbb{R}^n \mid -aG^{-1}(2a/b) < x_1 < 0\}$ locally uniformly in x'.*

To treat a multi-dimensional traveling spot, we divide it into three part: a front, a back, and a tip. The interface where $V > 0$ (resp. $V < 0$, $V = 0$) is called a *front* (resp. a *back*,



(a) $c = 0.6,\ b = 0.31685$ (b) $c = 0.85,\ b = 0.63448$ (c) $c = 0.95,\ b = 0.8003178$

FIGURE 2. Profiles of the boundaries of Ω_c of 3D traveling spots when $n = 3,\ g_1 = 1,\ g_2 = 44,\ g_3 = 100,\ g_4 = 1$, and $a = 1$.

a *tip*). Compared with the work [5] when $n = 2$, a singularity appears in this case (see more detail in (2.2)). It makes the analysis become more complicated even for the front (see Section 3). Since the proof of the existence of the back is based on the continuous dependence for the solution with respect to the parameter b, the method is not directly applicable to our case due to the singularity. To overcome this difficulty, we construct a series of backs on the approximate intervals. Then we show the existence of the back as its limit. In addition, we prove the convergence of the traveling spots to the planar traveling wave as c tends to a. This result implies that the traveling spots connect a stationary solution and the planar traveling wave.

The remainder of this paper is organized as follows. First, we derive the equations to describe the front and the back of the traveling spot in Section 2. Next, in Section 3, we show the existence and uniqueness of the front. Then, we prove the existence and uniqueness of the back in Section 4. Finally, we study the relationship between the shape of the spot and the speed c in Section 5. We show the convexity of the traveling spot for small c and that the traveling spot is close to the planer wave when c is close to a.

2. Problem setting

First we change the Euclidean coordinates (x_1, x') to cylindrical coordinates (x_1, r, ω) by the relation $r = |x'|$ and $x' = r\omega$. We assume that $\Omega(t)$ and $v(x_1, x', t)$ are radially symmetric in x'. Moreover, we assume

$$\begin{aligned} &\Omega(t) = \{(x_1, r\omega) \in \mathbb{R}^n \mid w_-(r) + ct \le x_1 \le w_+(r) + ct,\ \omega \in S^{n-2}\}, \\ &v(x_1, x') = \widetilde{v}(x_1, r), \end{aligned} \tag{2.1}$$

with some functions $w_\pm$. If the interface $\partial\Omega$ is represented by the graph $x_1 = w(r)$, then the normal vector of the curve $x_1 = w(r)$ in the (x_1, r)-plane is given by $\pm(-1, w_r)/\sqrt{w_r^2 + 1}$. We denote the angle between the outer normal vector and the x_1-axis in the (x_1, r)-plane by θ. The unit normal vector N in the (x_1, x')-plane can be written as

$$N = \left(\cos\theta, \frac{x'}{|x'|} \sin\theta \right).$$

Notice that θ only depends on r. We compute that

$$\begin{aligned} \mathrm{div} N &= \frac{\partial}{\partial x_1} \cos\theta + \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \left(\frac{x_j}{|x'|} \sin\theta \right) \\ &= \frac{n-1}{r} \sin\theta - \sum_{j=2}^{n} \frac{x_j^2}{r^3} \sin\theta + \sum_{j=2}^{n} \frac{x_j^2}{r^2} \cos\theta \frac{d\theta}{dr} \\ &= \frac{n-2}{r} \sin\theta + \cos\theta \frac{d\theta}{dr}. \end{aligned}$$

By the definition of a mean curvature, we have

$$\kappa = \frac{1}{n-1} \mathrm{div} N = \frac{1}{n-1} \left((n-2) \frac{\sin\theta}{r} + \frac{d\theta}{dr} \cos\theta \right).$$

Thus we derive

$$\frac{d\theta}{dr} = \frac{(n-1)\kappa - (n-2)\dfrac{\sin\theta}{r}}{\cos\theta}.$$

Introducing the arc length s of the curve in the (x_1, r)-plane and combining (1.8), we obtain the following system to describe traveling spots:

$$\left\{ \begin{aligned} \frac{dx_1}{ds} &= -\sin\theta, \\ \frac{dr}{ds} &= \cos\theta, \\ \frac{d\theta}{ds} &= W(v) - c\cos\theta - (n-2)\frac{\sin\theta}{r}, \\ -c\frac{\partial v}{\partial x_1} &= g(\chi_{\Omega(0)}, v). \end{aligned} \right. \tag{2.2}$$

We remark that x_1, r, θ, v and c are unknown and that the interface will be determined by the first three equations for any c if v is given on it. We recall that the interface where $V > 0$ (resp. $V < 0$, $V = 0$) is denoted by a front (resp. a back, a tip). We denote the front and the back solutions of the first three equations by

$$(x_{1,+}, r_+, \theta_+), \quad (x_{1,-}, r_-, \theta_-)$$

respectively for a given function v and a constant c. Note that the graph $x_1 = w_+(|x'|)$ (resp. $x_1 = w_-(|x'|)$) corresponds to the front (resp. the back). We will specify $(x_{1,+}, r_+, \theta_+)$, $(x_{1,-}, r_-, \theta_-)$ later.

Remark 1. When $n = 2$, the last term of the equation for θ in (2.2) vanishes. The equation of $d\theta/ds$ may have a singularity at $r = 0$ when $n \geq 3$. This causes the new difficulty to show the existence of the front and the back.

3. Existence and uniqueness of the front

We may assume that $v = 0$ on

$$\Omega^+(t) = \{(x_1, r\omega) \in \mathbb{R}^n \mid x_1 \geq w_+(r) + ct,\ \omega \in S^{n-2}\}.$$

Especially $v = 0$ on the front. Thus (2.2) becomes

$$\left\{\begin{aligned} \frac{dx_{1,+}}{ds} &= -\sin\theta_+, \\ \frac{dr_+}{ds} &= \cos\theta_+, \\ \frac{d\theta_+}{ds} &= a - c\cos\theta_+ - (n-2)\frac{\sin\theta_+}{r}. \end{aligned}\right. \tag{3.1}$$

Note that $w_+(r_+(s)) = x_{1,+}(s)$ as long as the solution of (3.1) exists. Due to the independence of the equations for r and θ on x_1, it turns out that the solution with some shift along the x_1-axis also satisfies (3.1). Thus, we can put the initial value as follows:

$$x_{1,+}(0) = 0, \quad r_+(0) = 0, \quad \theta_+(0) = 0. \tag{3.2}$$

To show the existence of the solution of the initial value problem (3.1)-(3.2) of the front, we first consider the condition for the $d\theta_+(0)/ds$. Using the l'Hospital rule, we have

$$\begin{aligned} \frac{d\theta_+}{ds}(0) &= a - c\cos\theta(0) - (n-2)\cos\theta(0)\frac{d\theta_+}{ds}(0) \\ &= a - c - (n-2)\frac{d\theta_+}{ds}(0), \end{aligned}$$

which implies

$$\frac{d\theta_+}{ds}(0) = \frac{a-c}{n-1} > 0.$$

The next lemma guarantees the existence and uniqueness of the local solution of (3.1)-(3.2).

Lemma 1. *For each $c \in (0, a)$, there exists a unique solution $(x_{1,+}, r_+, \theta_+)(s)$ of the system* (3.1) *with* (3.2) *such that*

$$\begin{aligned} x_{1,+}(s) &= -\frac{a-c}{2(n-1)s^2} + O(s^3), \\ r_+(s) &= s + O(s^2), \\ \theta_+(s) &= \frac{a-c}{n-1}s + O(s^2) \end{aligned}$$

for $0 \leq s \leq s_0$ where s_0 is sufficiently small.

This lemma can be shown by the similar argument to Guo-Ninomiya-Tsai [9, Lemma 3.1]. So, we omit the proof.

When $\theta_s > 0$, s can be written as the function of θ. Set $R(\theta) := r_+(s(\theta))$, then $R(\theta)$ satisfies the following equations:

$$\left\{\begin{aligned} &\frac{dR}{d\theta} = \frac{R\cos\theta}{(a - c\cos\theta)R - (n-2)\sin\theta}, \\ &R(0) = 0, \quad \frac{dR}{d\theta}(0) = \frac{n-1}{a-c}. \end{aligned}\right. \tag{3.3}$$

By Lemma 1, the local solution $R(\theta)$ exists and is positive for $\theta \in (0, \theta(s_0)]$.

Lemma 2. *The following inequality holds:*

$$R(\theta) \geq \frac{(n-1)\sin\theta}{a - c\cos\theta},$$

as long as the solution of (3.3) *exists.*

Proof. A simple calculation leads us to

$$\begin{aligned}\frac{d}{d\theta}\left(R - \frac{(n-1)\sin\theta}{a - c\cos\theta}\right) &= \frac{(n-2)\cos\theta}{(a - c\cos\theta)R - (n-2)\sin\theta}\left(\frac{(n-1)\sin\theta}{a - c\cos\theta} - R\right) \\ &\quad + \frac{(n-1)c\sin^2\theta}{(a - c\cos\theta)^2}.\end{aligned}$$

This lemma immediately follows from the Grownwall inequality. □

Thus the solution of (3.3) can be extended for any $\theta \in [0, \pi/2]$. Since $dx_{1,+}/ds$ is monotone increasing in $\theta \in (0, \pi/2)$, the solution of the initial value problem (3.1)-(3.2) also exists uniquely by Lemmas 1 and 2.

Now we denote the size of the front by $r_0(c) := R(\pi/2)$.

Proposition 1. *The function $r_0(c)$ is strictly increasing in $c \in (0, a)$. Furthermore, we have $r_0(c) \to +\infty$ as $c \nearrow a$.*

Proof. By (3.3) we have

$$\left(\frac{\partial R}{\partial c}\right)_\theta = A(\theta)\left\{R^2\cos^2\theta - (n-2)\frac{\partial R}{\partial c}\sin\theta\cos\theta\right\},$$

where

$$A(\theta) := \frac{1}{\left\{(a - c\cos\theta)R - (n-2)\sin\theta\right\}^2}.$$

Using the Grownwall inequality, we see that $(\partial R/\partial c)(\theta) \geq 0$. Then we compute that

$$\begin{aligned}\frac{\partial R}{\partial c}(\theta) = \int_0^\theta &\left[A(\tilde\theta)R^2(\tilde\theta)\cos^2\tilde\theta \exp\left(\int_0^{\tilde\theta}(n-2)A(\hat\theta)\sin\hat\theta\cos\hat\theta d\hat\theta\right)\right]d\tilde\theta \\ &\times \exp\left(-\int_0^\theta (n-2)A(\tilde\theta)\sin\tilde\theta\cos\tilde\theta d\tilde\theta\right).\end{aligned}$$

If $(\partial R/\partial c)(\pi/2) = 0$, from the last equality, we know that $R(\theta) = 0$ for all $\theta \in [0, \pi/2]$ and get a contradiction. Therefore, $(\partial R/\partial c)(\pi/2) > 0$ and we get the conclusion.

For $\theta \in (0, \pi/2)$, it easy to see that

$$\frac{dR}{d\theta} = \frac{R\cos\theta}{(a - c\cos\theta)R - (n-2)\sin\theta} \geq \frac{\cos\theta}{a - c\cos\theta}.$$

Integrating both sides from 0 to $\pi/2$, we obtain that

$$r_0(c) \geq -\frac{\pi}{2c} + \frac{2a}{c\sqrt{a^2 - c^2}}\tan^{-1}\left(\frac{a+c}{\sqrt{a^2 - c^2}}\right).$$

Since the right hand side of the last inequality tends to infinity as c tends to a, we get the conclusion. □

Proposition 2. *When $c = 0$, the front is a semi-circle. Namely,*

$$x_{1,+}(s) = \frac{1}{a}(\cos(as) - 1), \quad r_+(s) = \frac{1}{a}\sin(as), \quad \theta_+(s) = as.$$

Furthermore, we have $r_0(0) = (n-1)/a$.

Proof. When $n \geq 3$, we can rewrite the first equality in (3.3) as follows:

$$(aR^{n-2} - (n-2)R^{n-3}\sin\theta)dR - R^{n-2}\cos\theta d\theta = 0$$

Since

$$\frac{\partial}{\partial\theta}(aR^{n-2} - (n-2)R^{n-3}\sin\theta) = -(n-2)R^{n-3}\cos\theta = \frac{\partial}{\partial R}(-R^{n-2}\cos\theta)$$

and $R(0) = 0$, it is easy to show that

$$R(\theta) = \frac{(n-1)\sin\theta}{a}.$$

This implies that $r_+(s) = (n-1)\sin\theta_+(s)/a$. Then we get the conclusion by substituting this into (3.1)-(3.2). $\square$

Remark 2. We know that $r_0(c) > (n-1)/a$ for every $c \in (0, a)$ by Propositions 1 and 2.

Recall that the solution of the front with some shift along the x_1-axis is also a solution. Thus we have shown the existence of the front satisfying (3.1) with

$$r_+(0) = 0 \text{ when } \theta_+(0) = 0, \tag{3.4}$$

$$x_{1,+}(s_0) = 0 \text{ when } \theta_+(s_0) = \frac{\pi}{2}. \tag{3.5}$$

Hereafter we always assume the above property for the front.

4. Existence and uniqueness of the back

Recall that $w_\pm$ is defined in (2.1). Since s can be represented as a function of r, we define $(w_+(r), \varphi_+(r)) = (x_{1,+}(s(r)), \theta_+(s(r))$. Then, the second equation of (1.8) becomes

$$-cv_{x_1} = g(\chi_{\Omega(0)}, v) = \begin{cases} g(1, v), & \text{if } w_-(r) \leq x_1 \leq w_+(r),\ |r| \leq r_0(c), \\ g(0, v), & \text{otherwise.} \end{cases}$$

Due to the symmetry assumption, v depends only on $x_1 - ct$ and $r = |x'|$. Hence we use $v(x_1 - ct, r)$ instead of $v(x_1 - ct, x')$. Since $v = 0$ on the front, we get

$$w_+(r) - x_1 = \int_0^{v(x_1, r)} \frac{c}{g(1,\xi)} d\xi. \tag{4.1}$$

We introduce two functions G_0 and G_1 defined by

$$G_1^{-1}(v) := \int_0^v \frac{d\xi}{g(1,\xi)}, \qquad G_0^{-1}(v) := \int_1^v \frac{d\xi}{g(0,\xi)}$$

respectively. If g is given by (1.6), then

$$\begin{aligned} G_0^{-1}(v) &= \frac{g_3}{g_2}(1 - v) - \frac{g_4}{g_2}\log v, \\ G_1^{-1}(v) &= \frac{g_3 v}{g_1 g_3 - g_2} - \frac{g_4 g_2}{(g_2 - g_1 g_3)^2}\log\frac{g_4 g_1 + (g_1 g_3 - g_2)v}{g_4 g_1} \end{aligned}$$

as in [5]. From the definition of G_1 and (4.1), we obtain

$$v(x_1, r) = G_1\left(\frac{w_+(r) - x_1}{c}\right)$$

for $w_-(r) \le x_1 \le w_+(r),\ 0 \le r \le r_0$.

Similar calculations induce

$$v(x_1, r) = \begin{cases} 0, & \text{if } x_1 \ge w_+(r) \text{ or } r \ge r_0, \\ G_1\left(\dfrac{w_+(r) - x_1}{c}\right), & \text{if } w_-(r) \le x_1 \le w_+(r),\ 0 \le r \le r_0, \\ G_0\left(G_0^{-1}(v_-(r)) + \dfrac{w_-(r) - x_1}{c}\right), & \text{if } x_1 \le w_-(r),\ 0 \le r \le r_0, \end{cases} \tag{4.2}$$

where

$$v_-(r) := G_1\left(\frac{w_+(r) - w_-(r)}{c}\right).$$

Using $w_+(r)$ and $v_-(r)$, we can derive the equation of the back as follows:

$$\begin{cases} \dfrac{dx_{1,-}}{ds} = -\sin\theta_-, \\ \dfrac{dr_-}{ds} = \cos\theta_-, \\ \dfrac{d\theta_-}{ds} = W\left(G_1\left(\dfrac{w_+(r_-) - x_{1,-}}{c}\right)\right) - c\cos\theta_- - (n-2)\dfrac{\sin\theta_-}{r_-} \end{cases} \tag{4.3}$$

with the initial condition

$$(x_{1,-}(0), r_-(0), \theta_-(0)) = \left(0, r_0(c), \frac{\pi}{2}\right). \tag{4.4}$$

The local existence and uniqueness of solutions of (4.3) and (4.4) will be guaranteed by Lemma 3. We denote the solution of (4.3) with (4.4) by $(x_{1,-}(s), r_-(s), \theta_-(s)) = (x_{1,-}, r_-, \theta_-)(s; c, b)$. By the assumption that the traveling spot is symmetric with respect to the x_1-axis, we look for the positive constants $b^* = b^*(c)$ and $s^* = s^*(c)$ such that the solution should satisfy the following conditions:

$$r_-(s^*; b^*) = 0, \quad \theta_-(s^*; b^*) = \pi. \tag{4.5}$$

Note that the third equation of the system (4.3) has the singularity when $r_-(s^*; b^*) = 0$.

4.1. **Basic properties.** To show the existence and uniqueness of a solution of (4.3)–(4.5), we prepare several properties of the local solution of (4.3) and (4.4).

Lemma 3. *For each $c \in (0,a)$ and each $b > 0$, there exists a unique solution $(x_{1,-}, r_-, \theta_-)(s)$ of the system (4.3) with (4.4) such that*

$$\begin{aligned} x_{1,-}(s) &= -s + O(s^2), \\ r_-(s) &= r_0(c) - \frac{1}{2}\left(a - \frac{n-2}{r_0(c)}\right) s^2 + O(s^3), \\ \theta_-(s) &= \frac{\pi}{2} + \left(a - \frac{n-2}{r_0(c)}\right) s + O(s^2) \end{aligned}$$

for $0 \le s \ll 1$.

Since the proof is similar to [9, Lemma 3.1], we omit it.

For the simplicity of notation, we ignore the subscript minus sign from now on. By the similar argument as in [5, Lemma 4.2], the next lemma can be shown.

Lemma 4. *For $b > 0$, the following statements hold.*

(i) *If there is a $s_* > 0$ such that $\theta(s_*) = 3\pi/2$ and $\pi/2 < \theta(s) < 3\pi/2$ for $0 < s < s_*$, then $\theta(s) > 3\pi/2$ for $s > s_*$ with $s - s_*$ small.*

(ii) *If there is a $s_* > 0$ such that $\theta(s_*) = \pi/2$ and $\pi/2 < \theta(s) < 3\pi/2$ for $0 < s < s_*$, then $\theta(s) < \pi/2$ for $s > s_*$ with $s - s_*$ small.*

Proof. If the assumption of (i) holds, then we have $d\theta/ds(s_*) \ge 0$. When $d\theta/ds(s_*) > 0$, it is done. Otherwise, if $d\theta/ds(s_*) = 0$, we have

$$\begin{aligned} \frac{d^2\theta}{ds^2}(s_*) &= \left\{ -bG_1'\Big(\frac{w_+ - x_1}{c}\Big)\frac{w_{+,r}r_s - x_{1,s}}{c} + c\frac{d\theta}{ds}\sin\theta - (n-2)\frac{\cos\theta \cdot \frac{d\theta}{ds} r - \frac{dr}{ds}\sin\theta}{r^2} \right\}\Bigg|_{s=s_*} \\ &= \frac{b}{c}\, G_1'\Big(\frac{w_+(r(s_*)) - x_1(s_*)}{c}\Big) > 0 \end{aligned}$$

and get the conclusion of (i).

By the similar argument as the above, the conclusion of (ii) holds. □

Next, we consider the following open region:

$$D := \mathbb{R} \times (0, r_0(c)) \times \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

For any $c \in (0,a)$ and $b > 0$, we define exit-lengths $\underline{s}$, $\overline{s}$, $S(c,b)$ and an exit-point $(x_e, r_e, \theta_e)(c,b)$ as follows:

Definition 1. (exit-lengths and exit-points)

(i) If there is a positive number $\underline{s}$ such that the orbit stays in D for $0 < s < \underline{s}$, $r(\underline{s}) > 0$, $\theta(\tau) < \pi/2$ for $0 < \tau - \underline{s} \ll 1$, then $S(c,b) = \underline{s}$ and $(x_e, r_e, \theta_e)(c,b) = (x(\underline{s}), r(\underline{s}), \pi/2)$;

(ii) if there is a positive number $\overline{s}$ such that the orbit stays in D for $0 < s < \overline{s}$, $r(\overline{s}) > 0$, $\theta(\tau) > 3\pi/2$ for $0 < \tau - \overline{s} \ll 1$, then $S(c,b) = \overline{s}$ and $(x_e, r_e, \theta_e)(c,b) = (x(\overline{s}), r(\overline{s}), 3\pi/2)$;

(iii) if there is a positive number $\hat{s}$ such that the orbit stays in R for $0 < s < \hat{s}$ and $r(\hat{s}) = 0$, then $S(c,b) = \hat{s}$ and $(x_e, r_e, \theta_e)(c,b) = (x(\hat{s}), 0, \theta(\hat{s}))$.

From Lemma 4, the definitions of exit-length and exit-point are well-defined.

When $s=0$, we have $d\theta/ds = a-(n-2)/r_0 > 0$ by $r_0 > (n-1)/a$. Then we can define the following.

Definition 2. For the solution (x_1, r, θ) of the system (4.3) with (4.4) on $[0, S(c,b))$, we denote by $\tilde{s}(c,b)$ the first positive s such that $d\theta/ds = 0$ holds. Then we also define $\tilde{r} := r(\tilde{s}; c, b)$, $\tilde{z} := r_0(c) - \tilde{r}$ and $\tilde{\theta} := \theta(\tilde{s}; c, b)$.

Set $z := r_0(c) - r$. Since r is strictly decreasing for $s \in (0, S)$, we can define $X(z) := x_1(s(z))$ and $\Theta(z) := \theta(s(z))$. Also, we set

$$z_e := r_0(c) - r_e, \quad X_+(z) := w_+(r_0(c) - z).$$

Then (X, Θ) satisfies the following differential equations

$$\begin{cases} \dfrac{dX}{dz} = \dfrac{\sin\Theta}{\cos\Theta}, \\[2ex] \dfrac{d\Theta}{dz} = -\dfrac{a - c\cos\Theta - bG_1\left(\dfrac{X_+(z)-X(z)}{c}\right) - (n-2)\dfrac{\sin\Theta}{r_0(c)-z}}{\cos\Theta} \end{cases} \tag{4.6}$$

with

$$X(0) = 0, \quad \Theta(0) = \frac{\pi}{2}. \tag{4.7}$$

The following lemmas can be shown by using the comparison principle with respect to the parameters b and c (see also [9, Lemma 3.3]).

Lemma 5. *Fix a $c > 0$. For $0 < b_1 < b_2$, the solutions $(X_i(z), \Theta_i(z)) := (X(z; c, b_i), \Theta(z; c, b_i))$ of (4.6) and (4.7) defined on $[0, z(S(c, b_i))]$, $(i = 1, 2)$ satisfy*

$$X_1(z) \geq X_2(z), \quad \Theta_1(z) \geq \Theta_2(z)$$

on $(0, \min\{z(S(c,b_1)), z(S(c,b_2), r_0(c))\})$ as long as $X_+(z) > X_1(z)$.

Lemma 6. *Fix a $b > 0$. For $0 < c_1 < c_2$, the solutions $(X_i(z), \Theta_i(z)) := (X(z; c_i, b), \Theta(z; c_i, b))$ of (4.6) and (4.7) defined on $[0, z(S(c_i, b))]$, $(i = 1, 2)$ satisfy*

$$X_2(z) \geq X_1(z), \quad \Theta_2(z) \geq \Theta_1(z)$$

on $(0, \min\{z(S(c_1,b)), z(S(c_2,b), r_0(c))\})$ as long as $X_+(z) > X_2(z)$.

Next, we borrow an idea from [6, Lemma 3.4-3.5] to give some properties between the front and the back. Now we construct a special solution to (4.3).

Proposition 3. *Given $b > 0$. Set*

$$(\hat{x}(s), \hat{r}(s), \hat{\theta}(s)) := \left(x_{1,+}(-s) - c\,G_1^{-1}\left(\frac{2a}{b}\right), r_+(-s), \theta_+(-s) + \pi\right), \quad s \geq 0.$$

Then $(\hat{x}, \hat{y}, \hat{\theta})$ becomes a solution of the system (4.3) defined for $s \in [0, s_+]$ where $r_+(-s_+) = 0$ and so

$$(\hat{X}, \hat{\Theta}) := \left(X_+(z) - c\,G_1^{-1}\left(\frac{2a}{b}\right), \Theta_+(z) + \pi\right)$$

is also a solution of (4.6).

Proof. Using $2a = bG_1(G_1^{-1}(2a/b)) = bG_1((x_{1,+}(-s) - \hat{x}(s))/c)$, we can easily check that $(\hat{x}, \hat{r}, \hat{\theta})$ is the solution of (4.3). □

The next lemma shows the relation between $\Theta_+(z)$ and $\Theta(z)$.

Lemma 7. *Fix a $c \in (0, a)$. For each $b > 0$, the solution $(x(s; c, b), r(s; c, b), \theta(s; c, b))$ of* (4.3) *and* (4.4) *satisfies one of the following properties :*

(i) $\pi/2 < \Theta(z) < \Theta_+(z) + \pi$ *for any* $z \in (0, z_e)$.

(ii) *There exists a* $z_p = z_p(b) \in (0, z_e)$ *such that* $\pi/2 < \Theta(z) < \Theta_+(z) + \pi$ *for any* $z \in (0, z_p)$ *and* $\Theta(z_p) = \Theta_+(z_p) + \pi$.

Moreover, in the case of (ii), *we have*

$$X_+(z) - X(z) < c\, G_1^{-1}\left(\frac{2a}{b}\right) \quad \textit{for} \ \ 0 < z \leq z_e,$$
$$\Theta(z) > \Theta_+(z) + \pi \quad \textit{for} \ \ z \in (z_p, z_e).$$

By using the similar argument as the proof in [5, Lemma 3.3], the conclusion of Lemma 7 can be shown.

Lemma 8. *For any $c \in (0, a)$, $\Theta(z_e(c, 0)) \in (\pi, 3\pi/2]$.*

Proof. If $z_e(c, 0) < r_0(0) < r_0(c)$, we have $\Theta(z_e(c, 0)) = 3\pi/2$ or $\pi/2$ by Definition 1. The comparison principle implies that

$$\Theta(z_e(c, 0)) = \Theta(z_e(c, 0); c, 0) > \Theta(z_e(c, 0); 0, 0) > \frac{\pi}{2}.$$

Hence, we know that $\Theta(z_e(c, 0)) = 3\pi/2$.

For the case $z_e(c, 0) \geq r_0(0)$, by the comparison principle, we have

$$\Theta(r_0(0); c, 0) > \Theta(r_0(0); 0, 0) = \pi.$$

Hence, there exists a $z_1 \in (0, r_0(0))$ such that $\Theta(z_1; c, 0) = \pi$. Now we claim that $\Theta(z; c, 0) > \pi$ for any $z \in (z_1, z_e(c, 0)]$. If not, there exists a $z_2 \in (z_1, z_e(c, 0)]$ such that $\Theta(z_2; c, 0) = \pi$ and $d\Theta(z_2; c, 0)/dz \leq 0$. But we also have

$$\frac{d\Theta(z_2; c, 0)}{dz} = a + c > 0$$

and get a contradiction. Therefore, we get the conclusion. □

From the next two lemmas, we know that $\Theta(z_e) = \pi/2$ for b sufficiently large. Although the proofs of the next two lemmas are similar to the proofs in [5, Lemma 5.2-5.3], for the reader's convenience, we provide the details here.

Lemma 9. *For each $\gamma \in (\pi/2, \pi]$, set*

$$\tilde{b} := \tilde{b}(\gamma, c) = \frac{a + c}{G_1(X_+(R(\gamma; c, 0))/c)}.$$

where $s_1 \in [0, \hat{s})$ satisfies $\theta(s_1; c, b) = \gamma$. If $b > \tilde{b}$, then

$$\frac{\pi}{2} < \tilde{\theta}(c, b) \leq \gamma.$$

Moreover,

$$\lim_{b \to +\infty} \tilde{\theta}(c, b) = \frac{\pi}{2}, \quad \lim_{b \to +\infty} \tilde{r}(c, b) = r_0(c). \tag{4.8}$$

Proof. Fix a $\gamma \in (\pi/2, \pi]$. Since $R(\gamma; c, 0)$ belongs to the interval $(0, r_0(c))$ by Lemma 8, $\tilde{b}$ is well defined. By the contradiction argument, we assume that there exist a $b > \tilde{b}$ and the smallest $s_1 > 0$ such that $\theta(s_1; c, b) = \gamma$. Then we know that $x_1(s_1) < 0$ and $\theta'(s_1; c, b) > 0$. By the comparison principle, we also have $r(s_1; c, b) > r(s_1; c, 0)$. From these results and the fact G_1 is increasing, we obtain that

$$\begin{aligned}\theta'(s_1) &= a - c\cos\gamma - bG_1((X_+(r(s_1; c, b)) - x_1(s_1; c, b))/c) - \frac{(n-2)\sin\gamma}{r(s_1; c, b)} \\ &< a + c - bG_1(X_+(r(s_1; c, b))/c) \\ &< a + c - bG_1(X_+(R(\gamma; c, 0))/c) \\ &< a + c - \tilde{b}G_1(X_+(R(\gamma; c, 0))/c) = 0.\end{aligned}$$

Hence, this contradicts the definition of s_1. So we have $\pi/2 < \tilde{\theta}(c, b) \leq \gamma$ for all $b > \tilde{b}$.

Since

$$\lim_{\gamma \to \pi/2} R(\gamma; c, 0) = r_0(c),$$

we have

$$\lim_{\gamma \to \pi/2} \tilde{b} = +\infty.$$

Therefore, (4.8) holds. □

Lemma 10. *For each $c \in (0, a)$, there is a positive constant $\underline{b} := \underline{b}(c)$ such that $S(c, b) = \underline{s}$ for all $b \geq \underline{b}$.*

Proof. By Lemma 9, for $\gamma = 3\pi/4$, we can take b_1 so large that

$$\tilde{\theta}(c, b) \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \quad \tilde{r}(c, b) \in \left(\frac{7}{8}r_0(c), r_0(c)\right)$$

for $b > b_1$. From this, we have

$$0 < \tilde{z} < \frac{r_0(c)}{8}, \quad \Theta'(z) > 0 \quad \text{for} \quad z \in (0, \tilde{z}).$$

When $\Theta'(z) = 0$ and $\Theta \in (\pi/2, \pi)$, we obtain that

$$\frac{d^2\Theta}{dz^2} = \frac{bG_1'\Big(\dfrac{X_+(z) - X(z)}{c}\Big)\dfrac{X_+'(z) - \tan\Theta}{c} + \dfrac{(n-2)\sin\Theta}{(r_0(c) - z)^2}}{\cos\Theta} < 0$$

holds. This also implies that $\Theta(z)$ is decreasing on $(\tilde{z}, z_e)$. So $\pi/2 < \Theta(z) < 3\pi/4$ holds for $z \in (\tilde{z}, z_e)$.

If we assume that $z_e \geq r_0(c)$, $\Theta(z)$ can be defined on $[0, r_0(c)]$ with

$$\Theta(z) \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right) \quad \text{for} \quad z \in [0, r_0(c)].$$

Hence, for $z \in [0, r_0(c)]$, we obtain $X(z) < 0$ by $X(0) = 0$ and

$$\frac{dX}{dz}(z) = \frac{\sin\Theta}{\cos\Theta} < 0.$$

Since G_1 is increasing, we know that $G_1((X_+(z) - X(z))/c) \geq G_1(X_+(z)/c)$. Thus, for $\Theta \in (\pi/2, 3\pi/4)$,

$$\begin{aligned}
\frac{d\Theta}{dz} &= -\frac{a - c\cos\Theta - bG_1\Big(\dfrac{X_+(z) - X(z)}{c}\Big) - \dfrac{(n-2)\sin\Theta}{r_0(c) - z}}{\cos\Theta} \\
&< \sqrt{2}\left(a - c\cos\Theta - bG_1\Big(\frac{X_+(z)}{c}\Big)\right) \\
&< \sqrt{2}(a + c - bG_1\Big(\frac{X_+(z)}{c}\Big)) \\
&< \sqrt{2}\left(a + c - b\min_{z\in[r_0(c)/4, r_0(c)/2]} G_1\Big(\frac{X_+(z)}{c}\Big)\right)
\end{aligned}$$

holds. For any $b \geq \underline{b}$ where

$$\underline{b} := \max\left\{ b_1, \frac{a + c + \dfrac{2\pi}{\sqrt{2}r_0(c)}}{\min_{z\in[r_0(c)/4, r_0(c)/2]} G_1\Big(\dfrac{X_+(z)}{c}\Big)} \right\},$$

we have

$$\frac{d\Theta}{dz} < -\frac{2\pi}{r_0(c)}.$$

Integrating from $r_0(c)/4$ to $r_0(c)/2$ on both sides yields

$$\Theta\left(\frac{r_0(c)}{2}\right) - \Theta\left(\frac{r_0(c)}{4}\right) < -\frac{2\pi}{r_0(c)}\left(\frac{r_0(c)}{2} - \frac{r_0(c)}{4}\right) = -\frac{\pi}{2}.$$

It contradicts $|\Theta(r_0(c)/2) - \Theta(r_0(c)/4)| < \pi/4$ by $\Theta(z) \in [\pi/2, 3\pi/4)$. From this, we show that $S(c, b) = \underline{s}$ for b sufficiently large. ☐

4.2. **Existence and uniqueness of a solution of** (4.3)–(4.5)**.** Now we define the following set:

$$B := \left\{ b > 0 \;\middle|\; \begin{array}{l} \text{There exists a constant } z_p = z_p(b) \in (0, z_e) \text{ such that} \\ \Theta(z_p) = \Theta_+(z_p) + \pi \text{ and} \\ \pi/2 < \Theta(z) < \Theta_+(z) + \pi \text{ for any } z \in (0, z_p) \end{array} \right\}.$$

Then we have the following property of B.

Lemma 11. *The set B is nonempty, bounded and open.*

Since the proof of this lemma can be done with an argument similar to [6, Proposition 3.7], we do not repeat it here.

Lemma 12. *For any sufficiently small $\delta > 0$, there exists a b_δ such that $\Theta(r_0(c) - \delta; b_\delta) = \pi$*

Proof. By Lemma 11, for any sufficiently small $\delta > 0$, there exist a $\underline{b}_\delta$ and a $z_{p,\delta} := z_p(\underline{b}_\delta)$ such that $\Theta(z_{p,\delta}; \underline{b}_\delta) = \Theta_+(z_{p,\delta}) + \pi > \pi$. Also, by Lemma 9 and 10, there exists a $\overline{b}_\delta$ such that $\Theta(r_0(c) - \delta; \overline{b}_\delta) = \pi/2$. By the continuous dependence on b at $r_0(c) - \delta$, there exists a $b_\delta \in (\underline{b}_\delta, \overline{b}_\delta)$ such that $\Theta(r_0(c) - \delta; b_\delta) = \pi$. ☐

Taking the monotone decreasing sequence $\{\delta_j\}$ such that $\delta_j \to 0$ as $j \to \infty$, we define b^* by the limit of b_{δ_j} as $j \to \infty$, i.e.

$$b^* := \lim_{j\to\infty} b_{\delta_j}. \tag{4.9}$$

Lemma 13. *For small $\eta_0 > 0$, take a positive integer J satisfying*

$$\delta_J < \frac{(n-2)\sin\eta_0}{2(a+c)}, \quad \Theta_+(r_0(c)-\delta_J) < \eta_0. \tag{4.10}$$

Fix an integer $j \geq J$. For any $\eta > \eta_0$, if there exist a $z_\eta \in (0, r_0(c))$ and a $s_\eta \in (0, S(c,b))$ such that

$$z(s_\eta) = z_\eta, \qquad |\Theta(z_\eta; b_{\delta_j}) - \pi| = \eta,$$

then

$$|\Theta(z; b_{\delta_j}) - \pi| \geq \eta$$

for any $z \geq z_\eta$. Moreover, $X(z; b_{\delta_j})$ is also bounded.

Proof. First, we show that $X(z)$ is bounded if $|\Theta(z; b_{\delta_j}) - \pi| < \eta$ for any $z \in [r_0(c) - \delta_j, z_e)$. Indeed, since

$$-\tan(\pi+\eta) = \tan(\pi-\eta) \leq \frac{dX}{dz}(z; b_{\delta_j}) = \tan\Theta(z; b_{\delta_j}) \leq \tan(\pi+\eta),$$

the integration of both sides from $z = r_0(c) - \delta_j$ to z implies that

$$|X(z; b_{\delta_j}) - X(r_0(c) - \delta_j; b_{\delta_j})| \leq \tan(\pi+\eta)(z - (r_0(c) - \delta_j)) \leq \delta_j \tan(\pi+\eta).$$

Then we know that $X(z)$ is bounded for $z \in [r_0(c) - \delta_j, z_e)$ with $|\Theta(z; b_{\delta_j}) - \pi| < \eta$.

Next, we claim that $\Theta(z; b_{\delta_j}) > \pi + \eta$ for $z > z_\eta$ and $X(z; b_\delta)$ is bounded, if $\Theta(z_\eta; b_{\delta_j}) = \pi + \eta$. From the last inequality of (4.10), there exists $z_p(b_{\delta_j}) \in (r_0(c) - \delta_j, z_\eta)$ such that

$$\Theta(z_p(b_{\delta_j}); b_{\delta_j}) = \Theta_+(z_p(b_{\delta_j})) + \pi.$$

By Lemma 7,

$$X_+(z) - X(z; b_{\delta_j}) < c\, G_1^{-1}\left(\frac{2a}{b_{\delta_j}}\right)$$

for $z_p(b_{\delta_j}) \leq z < z_e$. From this, we obtain that

$$\begin{aligned}
\frac{d\Theta}{dz}(z; c, b_{\delta_j}) &= -\frac{1}{\cos\Theta}\left(a - b_{\delta_j} G_1\Big(\frac{X_+(z) - X(z)}{c}\Big) - c\cos\Theta - \frac{(n-2)\sin\Theta}{r_0(c) - z}\right) \\
&\geq -\frac{1}{\cos\Theta}\left(a - b_{\delta_j}\frac{2a}{b_{\delta_j}} - c - \frac{(n-2)\sin(\pi+\eta)}{\delta_j}\right) \\
&= -\frac{1}{\cos\Theta}\left(\frac{(n-2)\sin\eta}{\delta_j} - (a+c)\right) \\
&\geq -\frac{1}{\cos\Theta}\left(\frac{(n-2)\sin\eta_0}{\delta_J} - (a+c)\right) \\
&= -\frac{1}{\cos\Theta}(a+c) > 0
\end{aligned}$$

as long as $\Theta(z; b_{\delta_j}) \in (\pi+\eta, 3\pi/2)$. Thus, it follows from $\Theta(z_\eta; b_{\delta_j}) = \pi+\eta$ that $\Theta(z; b_{\delta_j}) > \pi+\eta$ for $z \in (z_\eta, z_e)$. Similarly, we have

$$\theta(s; b_{\delta_j}) - \theta(s_\eta; b_{\delta_j}) \geq (s - s_\eta)(a + c).$$

By

$$\theta(s; b_{\delta_j}) - \theta(s_\eta; b_{\delta_j}) \leq \frac{3\pi}{2} - (\pi + \eta) = \frac{\pi}{2} - \eta,$$

we derive that

$$s - s_\eta \leq \frac{\pi/2 - \eta}{a + c}.$$

Since $|dx/ds| = |-\sin\theta| \leq 1$, by integrating from $s = s_\eta$ to s, we obtain that

$$\begin{aligned} |x(s; b_{\delta_j})| &\leq |x(s_\eta; b_{\delta_j})| + |s - s_\eta| \\ &\leq |x(s_\eta; b_{\delta_j})| + \left|\frac{\pi/2 - \eta}{a + c}\right|. \end{aligned}$$

Hence, $x(s; b_{\delta_j})$ is bounded.

By a similar argument to the case $\theta(s_\eta; b_{\delta_j}) = \pi + \eta$, if $\theta(s_\eta; b_{\delta_j}) = \pi - \eta$, we obtain that $\Theta(z; b_{\delta_j}) < \pi - \eta$ for $z > z_\eta$ and $X(z; b_{\delta_j})$ is bounded. □

Lemma 14.

$$\lim_{\delta \to +0} \Theta(r_0(c) - \delta; b^*) = \pi$$

holds.

Proof. We show this lemma by a contradiction argument. Assume that there exist a $\eta > 0$ and a $z_{2\eta}$ such that

$$|\Theta(z_{2\eta}; b^*) - \pi| = 2\eta \tag{4.11}$$

Here $\hat{\delta}$ is defined by $z_{2\eta} = r_0(c) - \hat{\delta}$.

Since $\Theta(z; b)$ is continuous with respect to b at $z = z_{2\eta}$, for any $\varepsilon \in (0, \eta)$, there exists a large integer j such that

$$|\Theta(z_{2\eta}; b_{\delta_j}) - \Theta(z_{2\eta}; b^*)| < \varepsilon \tag{4.12}$$

holds. By (4.11) and (4.12), we get

$$|\Theta(z_{2\eta}; b_{\delta_j}) - \pi| > 2\eta - \varepsilon > \eta$$

From Lemma 13, for $z \geq z_{2\eta}$, we know that

$$|\Theta(z; b_{\delta_j}) - \pi| > \eta$$

By the fact $\Theta(r_0(c) - \delta_j; b_{\delta_j}) = \pi$, we have

$$r_0(c) - \delta_j \leq r_0(c) - \hat{\delta}$$

which implies that $\delta_j \geq \hat{\delta} > 0$. Then we get a contradiction by $\delta_j \to 0$ as $j \to \infty$. Therefore, the proof of this lemma has been completed. □

Proof of Theorem 1. The existence of the front has been shown in Section 3. For the back, it is shown that there exists a b^* given by (4.9) such that the corresponding solution of (4.3) satisfies the conditions (4.4) and (4.5). These results show the existence of Ω for any $c \in (0, a)$. To show that $\partial\Omega$ is C^2, we only need to treat it at the tip. Since

$$
\begin{aligned}
& x_{1,+}(s_0) = 0 = x_{1,-}(0), \quad r_+(s_0) = r_0(c) = r_-(0) \\
& \frac{dx_{1,+}}{ds}(s_0) = -1 = \frac{dx_{1,-}}{ds}(0), \quad \frac{dr_+}{ds}(s_0) = 0 = \frac{dr_-}{ds}(0), \\
& \frac{d^2x_{1,+}}{ds^2}(s_0) = 0 = \frac{d^2x_{1,-}}{ds^2}(0), \quad \frac{d^2r_+}{ds^2}(s_0) = -a + \frac{n-2}{r_0(c)} = \frac{d^2r_-}{ds^2}(0),
\end{aligned}
$$

we see that $\partial\Omega$ is C^2. For the uniqueness of b^*, by the fact $X_+(z) - X(z; b^*) > 0$ for all $z \in (0, r_0(c)]$, it follows from Lemma 5

From the formula of $v(x_1, r)$ in (4.2), it is easy to see that $v \in C^1(\mathbb{R}^n \setminus \partial\Omega) \cap C^0(\mathbb{R}^n)$ and $v(x_1, x') = 0$ if $|x'| \geq r_0$ or $x_1 \geq w_+(r)$. To show (1.9), we only need to consider the case x_1 tends to $-\infty$. When $x_1 \to -\infty$, we have $G_0^{-1}(v) \to \infty$. This implies that v tends to 0 as $x_1 \to -\infty$. Therefore, we get the conclusion. □

5. Properties of traveling spots

In this section we consider the profiles of traveling spots. In order to study them, we prepare the following lemmas.

Lemma 15. *The back of a traveling spot never intersects the front except for the tip* $(X_+(0), r_0(c))$.

Proof. Applying Proposition 7 to the back of traveling spots, we have

$$\frac{\pi}{2} < \Theta(z) < \Theta_+(z) + \pi \tag{5.1}$$

for any $0 < z < r_0(c)$. Then we derive that $(X_+ - X)(z)$ is increasing in $z \in (0, r_0(c))$ by $\Theta_+ \in (0, \pi/2)$, $\Theta \in (\pi/2, 3\pi/2)$ and

$$\frac{d(X_+ - X)(z)}{dz} = \frac{\sin(\Theta_+ - \Theta)}{\cos\Theta_+ \cos\Theta} > 0.$$

Hence we conclude to the non-existence of self-intersection points of traveling spots. □

Lemma 16. *For the back of a traveling spot,* Θ *can attain* π *at most once in* $z \in (0, r_0(c))$. *Moreover, if* $\Theta(z_q) = \pi$, *then*

$$\frac{d\Theta}{dz}(z_q) \geq 0. \tag{5.2}$$

Proof. First we note that $\Theta(r_0(c)) = \pi$. If there are more than two roots of $\Theta(z) = \pi$, we can assume that there are z_1 and z_2 $(0 < z_1 < z_2 \leq r_0(c))$ such that $\Theta(z_1) = \Theta(z_2) = \pi$ and $\Theta(z) < \pi$ for any $z \in (z_1, z_2)$. Then there is a $z_3 \in (z_1, z_2)$ such that

$$d\Theta/dz(z_3) = 0, \quad d^2\Theta/dz^2(z_3) \geq 0, \quad \frac{\pi}{2} < \Theta(z_3) < \pi.$$

On the other hand, we compute that

$$\begin{aligned}\frac{d^2\Theta}{dz^2} &= -\frac{\sin\Theta}{\cos^2\Theta}\frac{d\Theta}{dz}\left\{a - c\cos\Theta - bG_1\left(\frac{X_+ - X}{c}\right) - (n-2)\frac{\sin\Theta}{r_0(c)-z}\right\} \\ &- \frac{1}{\cos\Theta}\left\{c\sin\Theta\frac{d\Theta}{dz} - bG_1'\left(\frac{X_+ - X}{c}\right)\frac{1}{c}\frac{d(X_+ - X)(z)}{dz}\right. \\ &\left. -(n-2)\left[\frac{\sin\Theta}{(r_0(c)-z)^2} + \frac{\cos\Theta}{r_0(c)-z}\frac{d\Theta}{dz}\right]\right\}\end{aligned}$$

Substituting $z = z_3$ to the above equality, we have

$$\frac{d^2\Theta}{dz^2}(z_3) = \left\{\frac{b}{c\cos\Theta(z_3)}G_1'\left(\frac{X_+ - X}{c}\right)\frac{d(X_+ - X)(z_3)}{dz} + \frac{n-2}{\cos\Theta(z_3)}\left[\frac{\sin\Theta(z_3)}{(r_0(c)-z_3)^2}\right]\right\} < 0$$

since $\Theta_+(z_3) \in (0, \pi/2)$, $\Theta(z_3) \in (\pi/2, \pi)$ and the functions G_1 and $X_+ - X$ are increasing. This contradicts the definition of z_3. Thus the first statement has been shown.

If there is a z_q such that $\Theta(z_q) = \pi$ and $d\Theta/dz(z_q) < 0$, then there is a $z_0(> z_q)$ close to z_q such that $\Theta(z_q) > \pi$. This implies that there is another root of $\Theta(z) = \pi$ which is less than z_q, namely, there are at least two roots. This contradicts the first statement. $\square$

Lemma 17. *As for the back of a traveling spot, $\Theta(z)$ is increasing while $\Theta(z) \in (\pi/2, \pi)$.*

Proof. We show this lemma by contradiction. Assume that there is a $z_1 \in (0, r_0(c))$ such that

$$\frac{d\Theta}{dz}(z_1) < 0, \quad \frac{\pi}{2} < \Theta(z_1) < \pi.$$

According to Lemma 16, $\Theta(z) \in (\pi/2, \pi)$ for any $z \in (0, z_1)$. Since $\Theta(r_0(c)) = \pi$, there is a $z_2 \in (z_1, r_0(c))$ such that

$$\frac{d\Theta}{dz}(z_2) = 0, \quad \frac{d^2\Theta}{dz^2}(z_2) \geq 0, \quad \frac{\pi}{2} < \Theta(z_2) < \pi.$$

As in the proof of Lemma 16, we have

$$\frac{d^2\Theta}{dz^2}(z_2) = \left\{\frac{b}{c\cos\Theta(z_2)}\left[G_1'\left(\frac{X_+ - X}{c}\right)\frac{d(X_+ - X)(z_2)}{dz}\right] + \frac{n-2}{\cos\Theta(z_2)}\left[\frac{\sin\Theta(z_2)}{(r_0(c)-z_2)^2}\right]\right\} < 0.$$

This contradicts the definition of z_2. Hence the proof is complete. $\square$

5.1. **Convexity.** In this study, we classify the traveling spot into two types as in [9]:

(i) $\theta'(s) > 0$ on $[0, s^*)$ (Convex type),
(ii) $\theta'(s)$ changes its sign in $[0, s^*)$ (Non-convex type).

First, for $c = 0$, we have $b^*(0) = 0$, $s^* = \pi/2a$ and the corresponding solution

$$(x_{1,-}, r_-, \theta_-)(s; 0, 0) = \Big(-\frac{1}{a}\sin as, \frac{1}{a}(\cos as), \pi/2 + as\Big)$$

for $s \in [0, s^*]$. Since we have the uniqueness of $b^*(c)$ for each $c \in (0, a)$ and the comparison principle Lemma 5, by the similar argument as in [9, Lemma 6.1-6.3], we have the following two lemmas.

Lemma 18. *$b^*(c)$ is continuous in $c \in (0, a)$. Moreover, if $0 < c_1 < c_2 < a$ and $\theta'(s; c_i, b^*(c_i)) > 0$ for all $s \in [0, s^*(c_i))$, $i = 1, 2$, then $b^*(c_1) < b^*(c_2)$.*

Lemma 19. $\lim_{c\to 0^+} b^*(c) = 0$.

By the last two lemmas, the continuity of $\theta'(s;c,b^*(c))$ with respect to s, c, $b^*(c)$ and the fact $\theta'(s;0,0)=a>0$, we have $\theta'(s;c,b^*(c))>0$ for all $s\in[0,s^*(c)]$ for the sufficiently small $c>0$. This completes the proof of Theorem 2 (i). Moreover, we have

$$\lim_{c\to 0}(x_{1,+},r_+,\theta_+)(s;c) \quad = (x_{1,+},r_+,\theta_+)(s;0)$$

and

$$\lim_{c\to 0}(x_{1,-},r_-,\theta_-)(s;c,b^*(c)) \quad = (x_{1,-},r_-,\theta_-)(s;0,0).$$

Therefore, Theorem 2 (iii) holds and the region Ω of the traveling spot converges to a disk with the radius $1/a$ as c tends to 0.

Now we will show that the traveling spot is non-convex when c is close to a. From the results shown in the above, there exists a $c_0\in(0,a)$ such that the traveling spot is convex for all $c\in(0,c_0]$. On the other hand, we set $W_+(\theta):=w_+(R(\theta))$ where $w_+(r)=x_{1,+}(s(r))$. Then $W_+(\theta)$ satisfies the following system.

$$\begin{cases} \dfrac{dW_+(\theta)}{d\theta} = \dfrac{-\sin\theta}{a-c\cos\theta-(n-2)\dfrac{\sin\theta}{R(\theta)}} \\[2ex] W_+(\pi/2)=0. \end{cases}$$

For $\theta\in[0,\pi/2]$, we have

$$\frac{dW_+(\theta)}{d\theta}\le\frac{-\sin\theta}{a-c\cos\theta} \tag{5.3}$$

Integrating both sides of (5.3) from 0 to $\pi/2$, we have

$$W_+(0;c)\ge\frac{1}{c}\log\frac{a}{a-c}.$$

Then it is easy to see that $W_+(0;c)$ tends to ∞ when c tends to a. Since $b^*(c_0)$ is bounded, there exists a $\underline{c}>c_0$ such that

$$W_+(0;c)>aG_1^{-1}\left(\frac{2a}{b^*(c_0)}\right)$$

for all $c\in[\underline{c},a)$. Now we claim that the traveling spot is of type non-convex for all $c\in[\underline{c},a)$. If not, there exists a $\hat{c}\in[\underline{c},a)$ such that $\theta'(s;\hat{c},b^*(\hat{c}))>0$, $\theta_-(s;\hat{c},b^*(\hat{c}))\in(\pi/2,\pi)$ and $x_{1,-}(s;\hat{c},b^*(\hat{c}))<0$ for all $s\in[0,s^*(\hat{c}))$. By Lemma 18, we also have $b^*(\hat{c})>b^*(c_0)$. Since G_1 is increasing, we derive that

$$a-\hat{c}\cos\theta_- - b^*(\hat{c})G_1\left(\frac{w_+(r_-)-x_{1,-}}{c}\right)-(n-2)\frac{\sin\theta_-}{r_-}<2a-b^*(c_0)G_1\left(\frac{w_+(r_-)}{a}\right)$$

Then we know that $\theta'(s;\hat{c},b^*(\hat{c}))<0$ near $s=s^*(\hat{c})$ and get a contradiction. Therefore, we proved that the traveling spot is non-convex when c is close to a. The proof of Theorem 2 (ii) has been done.

5.2. **Convergence to a planar wave.** Now, it remains to show Theorem 2 (iv) holds. When $c = a$, it is easy to see that $(x_{1,+}, r_+, \theta_+)(s) = (0, s, 0)$ is the planar wave solution of (3.1)-(3.2) for any $s \geq 0$. So we have $\varphi_+(r) = 0$ for any $r \geq 0$. By Lemma 18, we define $b^*(a) := \lim_{c \nearrow a} b^*(c)$. For convenience, we denote $b^*(a)$ by b^*. For $b = b^*$ and $c = a$, we also derive that that $(X, \Theta)(z) = (X_+(z) - aG_1^{-1}(2a/b^*), \pi)$ is the planar solution of (4.6) for all $z \geq 0$ by

$$\frac{dX}{dz} = 0 = \frac{\sin \pi}{\cos \pi},$$

$$\frac{d\Theta}{dz} = 0 = -\frac{a - a\cos\pi - b^* G_1\left(\dfrac{X_+(z) - (X_+(z) - aG_1^{-1}(2a/b^*))}{a}\right) - (n-2)\dfrac{\sin\pi}{r_0(c) - z}}{\cos\pi}.$$

To show that the front converges to the planar wave as $c \nearrow a$, we claim that for any $L > 0$, $\varphi_+(r) = \Theta(r_0(c) - r)$ converges to 0 uniformly in $|r| < L$ as $c \nearrow a$. Lemma 2 implies that

$$r > k(\sin \varphi_+(r); c)$$

where

$$k(s; c) := \frac{s}{a - c\sqrt{1 - s^2}}.$$

Since the function k is monotone increasing in $s \in (0, \sqrt{1 - (c/a)^2})$, we can define the inverse function k^{-1} on $(0, 1/\sqrt{a^2 - c^2})$. Then, we have

$$\sin \varphi_+(r) < k^{-1}(r; c) < \sqrt{1 - \frac{c^2}{a^2}}$$

for $0 < r < 1/\sqrt{a^2 - c^2}$. When $c \nearrow a$, $k^{-1}(r; c)$ tends to 0 for any $r > 0$. Hence we get the conclusion. This means that the front converges to the planar wave locally uniformly as $c \nearrow a$, i.e.,

$$\lim_{c \nearrow a} X_+(z) = 0. \tag{5.4}$$

Next we consider the back. Now we want to show that

$$X_+(z) - X(z) - a\, G_1^{-1}\left(\frac{2a}{b^*}\right) \to 0 \tag{5.5}$$

as $c \nearrow a$. Since

$$-\frac{d\theta}{dr}(0) = a - b^* v(X(r_0(c)), 0) + c - (n-2)\lim_{r \to 0} \frac{\sin \theta}{r},$$

we have

$$-\frac{d\theta}{dr}(0) = \frac{a - b^* v(X(r_0(c)), 0) + c}{n - 1}.$$

As seen in the previous subsection, if c is sufficiently close to a, $\theta'(s)$ becomes negative near $r = 0$, namely,

$$-\frac{d\theta}{dr}(0) = \frac{a - b^* v(X(r_0(c)), 0) + c}{n - 1} < 0,$$

which yields

$$v(X(r_0(c)), 0) = G_1\left(\frac{X_+(r_0(c)) - X(r_0(c))}{c}\right) > \frac{a + c}{b^*}.$$

This immediately implies that

$$c\,G_1^{-1}\left(\frac{a+c}{b^*}\right) \le X_+(r_0(c)) - X(r_0(c)) \le c\,G_1^{-1}\left(\frac{2a}{b^*}\right).$$

Then we have

$$\lim_{c\nearrow a}(X_+(r_0(c)) - X(r_0(c))) = a\,G_1^{-1}\left(\frac{2a}{b^*}\right).$$

Next we show (5.5) by a contradiction argument in the interval $[r_0(c)-L, r_0(c))$. For $c \nearrow a$, there are $\delta > 0$ and $z_c \in [r_0(c)-L, r_0(c))$ such that

$$X_+(z_c) - X(z_c) \le c\,G_1^{-1}\left(\frac{2a-\delta}{b^*}\right). \tag{5.6}$$

Recall that the traveling spot is non-convex if c is close to a. Lemma 16 implies that there is a unique root z_q such that $\Theta(z_q) = \pi$ and $\Theta(z) > \pi$ for any $z \in (z_q, r_0(c))$. Moreover, we have

$$X_+(z_q) - X(z_q) < c\,G_1^{-1}\left(\frac{a+c}{b^*}\right)$$

by

$$\frac{d\Theta}{dz}(z_q) = a + c - b^*\,G_1\left(\frac{X_+(z_q) - X(z_q)}{c}\right) > 0.$$

For any $z \in (z_q, r_0(c))$,

$$X(z) < X(r_0(c)).$$

It turns out from (5.4) that, for $z_q < z < r_0(c)$,

$$a\,G_1^{-1}\left(\frac{2a}{b^*}\right) \ge X_+(z) - X(z) \ge X_+(z) - X(r_0(c)) \to a\,G_1^{-1}\left(\frac{2a}{b^*}\right) \tag{5.7}$$

as $c \nearrow a$. Thus by the definition of z_c, $r_0(c) - L \le z_c \le z_q$. By Lemma 17, we have

$$\frac{\pi}{2} \le \Theta(z) < \Theta(z_c) < \pi$$

for $0 \le z < z_c$. Thus

$$X(z_c) < X(0) = X_+(0).$$

On the other hand, for the front, we have

$$\begin{aligned}\frac{dx_+}{d\theta_+} &= -\frac{\sin\theta_+}{a - c\cos\theta_+ - \dfrac{(n-2)\sin\theta_+}{r_0(c)-z}} \\ &\le -\frac{\sin\theta_+}{a - c\cos\theta_+}.\end{aligned}$$

This implies

$$X_+(0) - X_+(z_c) \le \left[-\frac{1}{c}\log\left(a - c\cos\theta\right)\right]_{\theta=\Theta_+(z_c)}^{\theta=\pi/2} = -\frac{1}{c}\log\frac{a}{a - c\cos\Theta_+(z_c)}.$$

Since $\Theta_+(z) \to 0$ for any $z \in [r_0(c)-L, r_0(c)]$ as $c \nearrow a$, we obtain that

$$X_+(z_c) - X_+(0) \to \infty$$

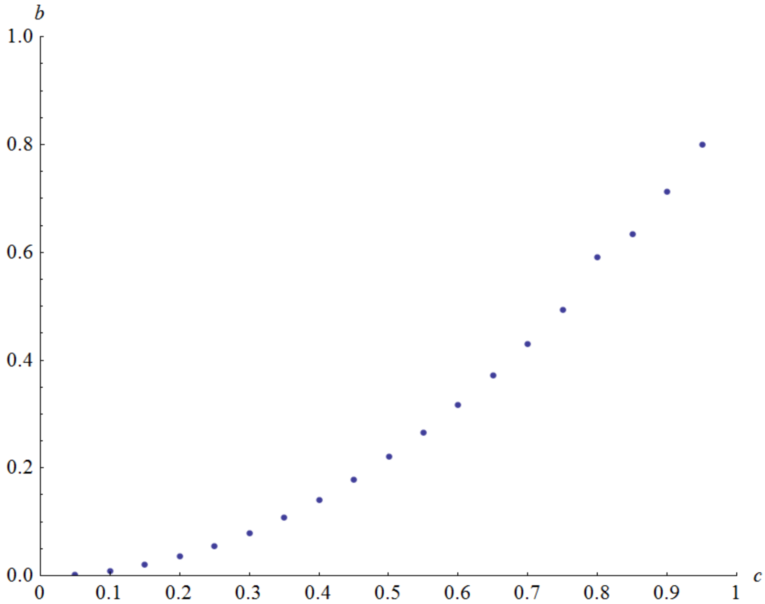


FIGURE 3. Relationship between c and b when $n = 3$, $g_1 = 1$, $g_2 = 44$, $g_3 = 100$, $g_4 = 1$ and $a = 1$.

as $c \nearrow a$. Therefore

$$X_+(z_c) - X(z_c) \geq X_+(z_c) - X_+(0) \to \infty.$$

This contradicts (5.6). Thus (5.5) has been shown. Now we give the proof of Theorem 2 (iv) as follows.

Proof of Theorem 2 (iv). From the results shown in the above, we show that for any $L > 0$, $\varphi_+(r) = \Theta(r_0(c) - r)$ converges to 0 uniformly in $|r| < L$ as $c \nearrow a$. This implies that the solution of the front $(x_{1,+}, r_+, \theta_+)(s; c)$ converges to the planar wave $(x_{1,+}, r_+, \theta_+)(s) = (0, s, 0)$. For the back, we have $X_+(z; c) - X(z; c)$ converges to $a\, G_1^{-1}(2a/b^*)$ as $c \nearrow a$. Therefore, we derive that the solution for the back $(X, \Theta)(z; c)$ converges to the planar wave $(X_+(z) - a\, G_1^{-1}(2a/b^*), \pi)$. $\square$

6. Conclusion

Traveling spot is one of the patterns observed in n-dimensional excitable media ($n \geq 3$). To capture this pattern we have used interface equations derived from the singular limit problem of the FitzHugh-Nagumo type equations, introduced in [5]. The traveling spot mainly consists of the front and the back. The system for the front is a set of three-component ordinary differential equations (3.1), (3.4), and (3.5). The system for the back is coupled with that for the front and three-component ordinary differential equations (4.3), (4.4), and (4.5). For $n \geq 3$, these systems include the singularities when $r = 0$. This makes the problem more complicated than the case [5] where $n = 2$. The existence of the traveling spot is shown by proving the existence of the front and the back if the speed c is smaller than that of a planar traveling wave. More precisely, for any $c \in [0, a)$, there is a constant

b such that the traveling spot uniquely exists up to the shift where a is a parameter of the system corresponding to the speed of a planar traveling wave. Unfortunately, we cannot show the existence of c for a given constant b. However, by numerical computations, the relationship between c and b seems to be monotone, see Figure 3. In addition, we classify the traveling spot into two cases: convex and non-convex. Furthermore, we obtain that the traveling spot is close to a planar wave when the wave speed c is close to a.

References

[1] M. Alfaro, D. Hilhorst, and H. Matano, *The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system*, Journal of Differential Equations **245** (2008), 505–565.

[2] M. Bode, A. W. Liehr, C. P. Schenk and H. G. Purwins, *Interaction of dissipative solitons: particle-like behaviour of localized structures in a three-component reaction-diffusion system*, Physica D: Nonlinear Phenomena, **161** (2002), 45-66.

[3] G. Carpenter, *A geometric approach to singular perturbation problems with applications to nerve impulse equations*, Journal of Differential Equations, **23**(1977), 335–367.

[4] X.-F. Chen, *Generation and propagation of interfaces in reaction-diffusion systems*, Transactions of the American Mathematical Society, **334** (1992), 877–913.

[5] Y.-Y. Chen, Y. Kohsaka and H. Ninomiya, *Traveling spots and traveling fingers in singular limit problems of reaction-diffusion systems*, DCDS-B **19(3)** (2014), 697–714.

[6] Y-Y. Chen, J.-S. Guo, and H. Ninomiya, *Existence and uniqueness of rigidly rotating spiral waves by a wave front interaction model.* Physica D: Nonlinear Phenomena **241** (2012), 1758–1766.

[7] B. Fiedler, A. Scheel, *Spatio-temporal dynamics of reaction-diffusion patterns*, In: Kirkilionis, M., Kromker, S., Rannacher, R., Tomi F. (eds.), Trends in Nonlinear Analysis, pp. 23–142, Berlin, Heidelberg, New York, Springer, 2003.

[8] P.C. Fife, *Understanding the patterns in the BZ reagent*, J. Statist. Phys. **39** (1985), 687–703.

[9] J.-S. Guo, H. Ninomiya and J.-C. Tsai, *Existence and uniqueness of stabilized propagating wave segments in wave front interaction model*, Physica D: Nonlinear Phenomena **239** (2010), 230–239.

[10] S. P. Hastings, *On the existence of homoclinic and periodic orbits for the FitzHugh-Nagumo equations*, The Quarterly Journal of Mathematics, **27** (1976), 123–134.

[11] D. Hilhorst, Y. Nishiura, and M. Mimura, *A free boundary problem arising in some reacting-diffusing system*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, **118** (1991), 355–378.

[12] J. Jalife, R.A. Gray, G.E. Morley and J.M. Davidenko, *Self-organization and the dynamical nature of ventricular fibrillation*, Chaos **8** (1998), 79–93.

[13] A. Karma, *Universal limit of spiral wave propagation in excitable media*, Physical review letters, **66** (1991), 2274.

[14] J.P. Keener and J.J. Tyson, *Spiral waves in the Belousov-Zhabotinskii reaction*, Physica D **21** (1986), 307–324.

[15] S. Koga, and Y. Kuramoto. *Localized patterns in reaction-diffusion systems*, Progress of Theoretical Physics **63** (1980), 106–121.

[16] K. Krischer, and A. Mikhailov. *Bifurcation to traveling spots in reaction-diffusion systems*, Physical review letters 73.23 (1994), 3165.

[17] W.F. Loomis, *The Development of Dioctyostelium Discoideum*, Academic Press, New York, 1982.

[18] E. Meron, *Pattern formation in excitable media*, Phys. Rep. **218** (1992), 1–66.

[19] A.S. Mikhailov, *Modeling pattern formation in excitable media: The Legacy of Norbert Wiener*, In: Milton, J., Jung P. (eds.), Epilepsy as a Dynamic Disease. Berlin,Heidelberg, New York, Springer, 2003.

[20] E. Mihaliuk, T. Sakurai, F. Chirila and K. Showalter, *Experimental and theoretical studies of feedback stabilization of propagating wave segments*, Faraday Discuss **120** (2001), 383–394.

[21] E. Mihaliuk, T. Sakurai, F. Chirila and K. Showalter, *Feedback stabilization of unstable propagating waves*, Phys. Review E. **65** (2002), 065602.

[22] H. Ninomiya and C.-H. Wu. *in preparation.*

[23] T. Ohta, M. Mimura, and R. Kobayashi. *Higher-dimensional localized patterns in excitable media*, Physica D: Nonlinear Phenomena **34** (1989), 115–144.

[24] P. Pelcé and J. Sun, *Wave front interaction in steadily rotating spirals*, Physica D: Nonlinear Phenomena **48** (1991), 353–366.

[25] J. Rinzel and J. B. Keller, *Traveling wave solutions of a nerve conduction equation*, **13** (1973) Biophysical Journal, 1313–1337.

[26] A. Scheel, *Bifurcation to spiral waves in reaction-diffusion systems*, SIAM J. Math. Anal., **29**, (1998), 1399-1418.

[27] J.P.Tyson and J. P. Keener, *Singular perturbation theory of traveling waves in excitable media (a review)*, Physica D: Nonlinear Phenomena **32** (1988), 327–361.

[28] V.S. Zykov and K. Showalter, *Wave front interaction model of stabilized propagating wave segments*, Phys. Review Letters **94** (2005), 068302.

Department of Mathematics, Tamkang University, 151, Yingzhuan Road, Tamsui, New Taipei City 25137, Taiwan

E-mail address: chenyanyu24@gmail.com

School of Interdisciplinary Mathematical Sciences, Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo 164-8525, Japan

E-mail address: ninomiya@math.meiji.ac.jp

UL Systems, Inc., Triton Square, Office Tower X 14F, 1-8-10 Harumi, Chuo-ku, Tokyo 104-6014, Japan

E-mail address: ryotaro.taguchi@ulsystems.co.jp

JEPE Vol 1, 2015, p. 307-323

Orthogonal $\perp$

ON THE ASYMPTOTIC BEHAVIOUR OF SOME PROBLEMS OF THE CALCULUS OF VARIATIONS

MICHEL CHIPOT

Abstract. In this note we analyse the asymptotic behaviour of the solution of some class of calculus of variation problems set in cylindrical domains. A special attention is given to limit the assumptions on the functional at stake to a minimum.

1. Monotonicity properties

We collect in this part some results which are perhaps known but for which we have been unable to find proper references. We refer the reader to [14], [15], [3], [4] for an introduction to the Sobolev spaces used here.
Let $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function such that

$$\mathcal{F}(v) = F(x, v(x), \nabla v(x)) \in L^1(\Omega)\ \forall v \in W_0^{1,q}(\Omega),\ q > 1. \tag{1.1}$$

For $f \in W^{-1,q'}(\Omega)$ the dual of $W_0^{1,q}(\Omega)$ we set

$$E_\Omega(v) = E_{\Omega,f}(v) = \int_\Omega \mathcal{F}(v)\ dx - \langle f, v\rangle\ \ \forall v \in W_0^{1,q}(\Omega) \tag{1.2}$$

where $\langle\ ,\ \rangle$ denotes the duality bracket between $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$. We consider the minimization problem of E_Ω on $W_0^{1,q}(\Omega)$ i.e. we denote by $u = u_f$ a function in $W_0^{1,q}(\Omega)$ such that

$$E_\Omega(u) \le E_\Omega(v)\ \forall v \in W_0^{1,q}(\Omega). \tag{1.3}$$

Then we have:

Theorem 1. *Let $u_1 = u_{f_1}$ and $u_2 = u_{f_2}$ be two minimizers of E_Ω on $W_0^{1,q}(\Omega)$ corresponding to f_1 and f_2 respectively. One has*

$$\left\langle f_1 - f_2, (u_1 - u_2)^+ \right\rangle \ge 0. \tag{1.4}$$

$(\ )^+$ *denotes the positive part of a function.*

2010 *Mathematics Subject Classification.* 49J40, 49J45, 35J20, 35J92.
Key words and phrases. Asymptotic behaviour, Calculus of variations, Minimizers, Cylindrical domains.
Received 24/12/2015, accepted 24/01/2016.
The research leading to these results has received funding from Lithuanian-Swiss cooperation programme to reduce economic and social disparities within the enlarged European Union under project agreement No CH-3-SMM-01/0.

Proof. Set

$$A = \{x \in \Omega \mid u_2(x) > u_1(x)\}. \tag{1.5}$$

Consider

$$v = u_1 \wedge u_2 = u_1 - (u_1 - u_2)^+.$$

One has - if A^C denotes the set $\Omega \backslash A$

$$E_{\Omega,f_1}(v) = \int_\Omega \mathcal{F}(v)\ dx - \langle f_1, v\rangle = \int_A \mathcal{F}(u_1)\ dx + \int_{A^C} \mathcal{F}(u_2)\ dx \\ - \langle f_1, u_1\rangle + \left\langle f_1, (u_1 - u_2)^+ \right\rangle .$$

We consider next

$$w = u_1 \vee u_2 = u_2 + (u_1 - u_2)^+.$$

One has

$$E_{\Omega,f_2}(w) = \int_\Omega \mathcal{F}(w)\ dx - \langle f_2, w\rangle = \int_A \mathcal{F}(u_2)\ dx + \int_{A^C} \mathcal{F}(u_1)\ dx \\ - \langle f_2, u_2\rangle - \left\langle f_2, (u_1 - u_2)^+ \right\rangle .$$

Adding up the last two equalities we get

$$\begin{aligned} &E_{\Omega,f_1}(v) + E_{\Omega,f_2}(w) \\ &\qquad = E_{\Omega,f_1}(u_1) + E_{\Omega,f_2}(u_2) + \left\langle f_1 - f_2, (u_1 - u_2)^+ \right\rangle . \end{aligned} \tag{1.6}$$

Thus (1.4) follows since u_1, u_2 are minimizers of E_{Ω,f_1}, E_{Ω,f_2} respectively. This completes the proof of the theorem.

□

Remark 1. Exchanging the role of f_1 and f_2 we have

$$\begin{aligned} \left\langle f_2 - f_1, (u_2 - u_1)^+ \right\rangle &= \left\langle f_2 - f_1, (u_1 - u_2)^- \right\rangle \\ &= \left\langle f_1 - f_2, -(u_1 - u_2)^- \right\rangle \geq 0 \end{aligned}$$

and thus adding to (1.4)

$$\langle f_1 - f_2, u_1 - u_2 \rangle \geq 0 \tag{1.7}$$

which means that the operator $f \to u_f$ is monotone.

Definition 1. We say that $f_1 \geq f_2$ when this inequality holds in the distributional sense i.e. when we have

$$\langle f_1, \varphi\rangle \geq \langle f_2, \varphi\rangle \ \ \forall \varphi \in \mathcal{D}(\Omega),\ \ \varphi \geq 0,$$

which by density of $\mathcal{D}(\Omega)$ in $W_0^{1,q}(\Omega)$ means

$$\langle f_1, v\rangle \geq \langle f_2, v\rangle \ \ \forall v \in W_0^{1,q}(\Omega),\ \ v \geq 0. \tag{1.8}$$

Then we have:

Corollary 1. *Let $u_1 = u_{f_1}$ and $u_2 = u_{f_2}$ be two minimizers of E_Ω on $W_0^{1,q}(\Omega)$ corresponding to f_1 and f_2 respectively. If*

$$f_1 \leq f_2 \tag{1.9}$$

one has

$$\left\langle f_1 - f_2, (u_1 - u_2)^+ \right\rangle = 0. \tag{1.10}$$

Proof. This follows immediately from the definition above and (1.4). $\square$

Remark 2. In the case where f_1 and f_2 are functions satisfying (1.9) one has $u_1 \leq u_2$ on the set where $f_1 < f_2$.

Corollary 2. *Let $u_1 = u_{f_1}$ and $u_2 = u_{f_2}$ be two minimizers of E_Ω on $W_0^{1,q}(\Omega)$ corresponding to f_1 and f_2 respectively. If (1.9) holds and either u_1 or u_2 is unique then*

$$u_1 \leq u_2. \tag{1.11}$$

Proof. Indeed in this case from (1.6) one derives

$$E_{\Omega,f_1}(v) - E_{\Omega,f_1}(u_1) = E_{\Omega,f_2}(u_2) - E_{\Omega,f_2}(w) = 0.$$

Then either $v = u_1$ or $w = u_2$ which in both cases means $u_1 \leq u_2$.

$\square$

As an immediate consequence one has

Corollary 3. *Suppose that 0 is solution to (1.3) for $f = 0$ and that (1.3) has a unique solution u_f for $f \geq 0$. Then one has $u_f \geq 0$.*

As a remarkable property we have also

Proposition 1. *The following assertions are equivalent:*

(i) The minimization problem (1.3) admits a unique solution.

(ii) If $u = u_f$ denotes a minimizer of E_Ω, the mapping $f \to u_f$ is monotone in the sense that

$$f_1 \leq f_2 \Longrightarrow u_{f_1} \leq u_{f_2}.$$

Proof.

•$(ii) \Longrightarrow (i)$.

Indeed if u_1 and u_2 are two solutions corresponding to the same f the monotonicity property implies $u_1 \geq u_2$ and $u_2 \geq u_1$ i.e. $u_1 = u_2$.

•$(i) \Longrightarrow (ii)$ follows directly from Corollary 2.

$\square$

Remark 3. The same results hold if the minimisation on $W_0^{1,q}(\Omega)$ is replaced by a minimisation on a subset K of $W_0^{1,q}(\Omega)$ having the property that

$$u_1, u_2 \in K \implies u_1 \wedge u_2,\ u_1 \vee u_2 \in K.$$

Note also that our results do not involve any convexity on F or K.

One has also the following additional monotonicity property with respect to the domain Ω.

Theorem 2. *Suppose that $f \geq 0$ in the distributional sense and that (1.3) admits a unique solution for two open subsets Ω and Ω' such that*

$$\Omega \subset \Omega'. \tag{1.12}$$

If u, u' denote the solutions to (1.3) corresponding to Ω, Ω' respectively and if 0 is solution to (1.3) for $f = 0$ for Ω, Ω' one has

$$u' \geq u \geq 0. \tag{1.13}$$

Proof. $u, u' \geq 0$ is a consequence of Corollary 3. Suppose that u is extended by 0 outside Ω. Consider then $v = u \wedge u' = u' - (u' - u)^+$ and

$$A = \{x \in \Omega' \mid u(x) > u'(x)\}.$$

Note that A is included in Ω. One has

$$\int_\Omega \mathcal{F}(v)\, dx - \langle f, v\rangle = \int_{A^C} \mathcal{F}(u)\, dx + \int_A \mathcal{F}(u')\, dx - \langle f, v\rangle .$$

If

$$\int_A \mathcal{F}(u')\, dx < \int_A \mathcal{F}(u)\, dx - \langle f, u\rangle + \langle f, v\rangle$$

one has

$$\begin{aligned}\int_\Omega \mathcal{F}(v)\, dx - \langle f, v\rangle &< \int_{A^C} \mathcal{F}(u)\, dx + \int_A \mathcal{F}(u)\, dx - \langle f, u\rangle + \langle f, v\rangle - \langle f, v\rangle \\ &= \int_\Omega \mathcal{F}(u)\, dx - \langle f, u\rangle .\end{aligned}$$

and a contradiction with the definition of u which is not a minimizer. Thus it holds

$$\int_A \mathcal{F}(u)\, dx \leq \int_A \mathcal{F}(u')\, dx + \langle f, u\rangle - \langle f, v\rangle .$$

If now one considers $w = u \vee u' = u + (u' - u)^+$ it comes

$$\int_{\Omega'} \mathcal{F}(w)\, dx - \langle f, w\rangle = \int_{A^C} \mathcal{F}(u')\, dx + \int_A \mathcal{F}(u)\, dx - \langle f, w\rangle$$

and thus

$$\begin{aligned}\int_{\Omega'} \mathcal{F}(w)\, dx - \langle f, w\rangle &\leq \int_{\Omega'} \mathcal{F}(u')\, dx + \langle f, u\rangle - \langle f, v\rangle - \langle f, w\rangle \\ &= \int_{\Omega'} \mathcal{F}(u')\, dx - \langle f, u'\rangle .\end{aligned}$$

(Recall that $v = u' - (u' - u)^+$, $w = u + (u' - u)^+$). By uniqueness of the solution one has $w = u'$ which implies that $u' \geq u$. This completes the proof of the theorem.

□

2. A convergence result

We denote by Ω_ℓ the open subset of $\mathbb{R}^n$ defined as

$$\Omega_\ell = \ell\omega_1 \times \omega_2$$

where

$$\omega_1 \text{ is a bounded convex open subset of } \mathbb{R}^p \text{ containing } 0, \tag{2.1}$$

ω_2 being a bounded open subset of $\mathbb{R}^{(n-p)}$.

Thus for every $\ell_0 \leq \ell - 1$ there exists a function $\rho = \rho_{\ell_0}(X_1)$ such that

$$0 \leq \rho \leq 1,\ \rho = 1 \text{ on } \ell_0\omega_1,\ \rho = 0 \text{ outside } (\ell_0 + 1)\omega_1,\ |\nabla\rho| \leq C \tag{2.2}$$

where C is a constant independent of ℓ_0. We consider then a function $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$F \quad \text{is convex,} \tag{2.3}$$

and such that there exist positive constants $\lambda, \Lambda, \Lambda'$ such that

$$\lambda|\xi|^q - \Lambda' \leq F(\xi) \leq \Lambda|\xi|^q + \Lambda' \ \forall \xi \in \mathbb{R}^n, \ F(0) = -\Lambda' \tag{2.4}$$

(q is a fixed number such that $q > 1$).

For such a function we have for some constant $C = C(q, \Lambda, \Lambda')$ (see [13]):

$$|F(P) - F(Q)| \leq C\{1 + |P|^{q-1} + |Q|^{q-1}\}|P - Q| \ \forall P, Q \in \mathbb{R}^n. \tag{2.5}$$

Then for

$$f = f(X_2) \in L^{q'}(\omega_2), \ \frac{1}{q} + \frac{1}{q'} = 1, \tag{2.6}$$

we consider the minimisation of the functional

$$E_{\Omega_\ell}(v) = \int_{\Omega_\ell} F(\nabla v) - fv \ dx \tag{2.7}$$

on $W_0^{1,q}(\Omega_\ell)$. We have:

Theorem 3. *There exists u_ℓ solution to*

$$u_\ell \in W_0^{1,q}(\Omega_\ell), \ E_{\Omega_\ell}(u_\ell) \leq E_{\Omega_\ell}(v) \ \forall v \in W_0^{1,q}(\Omega_\ell). \tag{2.8}$$

Moreover if F is strictly convex the solution of the problem above is unique.

Proof. First note that by (2.4) and (2.6), $E_{\Omega_\ell}(v)$ is well defined for any $v \in W_0^{1,q}(\Omega_\ell)$. Moreover by the Hölder and Poincaré inequalities one has

$$\int_{\Omega_\ell} fv \ dx \leq |f|_{q',\Omega_\ell}|v|_{q,\Omega_\ell} \leq C|f|_{q',\Omega_\ell}\big||\nabla v|\big|_{q,\Omega_\ell} \tag{2.9}$$

and thus -see (2.4)-

$$E_{\Omega_\ell}(v) \geq \lambda\big||\nabla v|\big|^q_{q,\Omega_\ell} - C|f|_{q',\Omega_\ell}\big||\nabla v|\big|_{q,\Omega_\ell} - \Lambda'|\Omega_\ell|.$$

This implies that E_{Ω_ℓ} is coercive or that every minimizing sequence is bounded in $W_0^{1,q}(\Omega_\ell)$. To conclude it is enough to show that E_{Ω_ℓ} is lower semicontinuous for the weak topology of $W_0^{1,q}(\Omega_\ell)$. E_{Ω_ℓ} being convex it is enough to show the lower semicontinuity of E_{Ω_ℓ} for the strong topology of $W_0^{1,q}(\Omega_\ell)$. If

$$v_n \to v \text{ in } W_0^{1,q}(\Omega_\ell)$$

up to a subsequence one has

$$v_n \to v \text{ in } L^q(\Omega_\ell), \ \nabla v_n \to \nabla v \text{ a.e. in } \Omega_\ell.$$

By Fatou's lemma one deduces

$$\begin{aligned}\liminf_{n\to\infty} E_{\Omega_\ell}(v_n) &= \liminf_{n\to\infty} \int_{\Omega_\ell} F(\nabla v_n) \ dx - \int_{\Omega_\ell} fv \ dx \\ &\geq \int_{\Omega_\ell} \liminf_{n\to\infty} F(\nabla v_n) \ dx - \int_{\Omega_\ell} fv \ dx = E_{\Omega_\ell}(v).\end{aligned}$$

This completes the proof of the theorem. □

We are interested here in the behaviour of u_ℓ when $\ell \to +\infty$. The issue has been considered before for a certain number of problems and we refer for instance to [1], [2], [5], [6], [7], [9], [10], [11] to have a good spectrum of the techniques involved.

If 0 denotes the 0 vector in $\mathbb{R}^p$ and ∇_{X_2} the gradient in X_2 i.e.

$$\nabla_{X_2} = (\partial_{x_{p+1}}, \cdots, \partial_{x_n})$$

one sets

$$E_{\omega_2}(v) = \int_{\omega_2} F(0, \nabla_{X_2} v) - fv \, dX_2 \ \forall v \in W_0^{1,q}(\omega_2). \tag{2.10}$$

Clearly $E_{\omega_2}(v)$ is well defined and with the same proof as above one has

Theorem 4. *There exists u_∞ solution to*

$$u_\infty \in W_0^{1,q}(\omega_2), \ E_{\omega_2}(u_\infty) \le E_{\omega_2}(v) \ \forall v \in W_0^{1,q}(\omega_2). \tag{2.11}$$

Moreover if $F(0, \cdot)$ is strictly convex the solution of the problem above is unique.

Our goal is now to show:

Theorem 5. *Suppose that $f \ge 0$ and F strictly convex then one has*

$$u_\ell \to u_\infty \tag{2.12}$$

when $\ell \to +\infty$.

Remark 4. For the time being we are not more precise on the way the convergence above occurs. This result completes the results of [8]. See also [5].

We will need several lemmas.

One denotes by $W^{1,q}_{lat}(\Omega_\ell)$ the space

$$W^{1,q}_{lat}(\Omega_\ell) = \{v \in W^{1,q}(\Omega_\ell) \mid v = 0 \text{ on } \ell\omega_1 \times \partial\omega_2\} \tag{2.13}$$

and by $V^{1,q}(\Omega_\ell)$ the space

$$V^{1,q}(\Omega_\ell) = \{v \in W^{1,q}_{lat}(\Omega_\ell) \mid ⨍_{\ell\omega_1} \nabla_{X_1} v \, dX_1 = 0 \text{ a.e. } X_2 \in \omega_2\}. \tag{2.14}$$

($⨍_{\ell\omega_1} = \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1}$ where $|\ell\omega_1|$ denotes the measure of $\ell\omega_1$).

Lemma 1. *u_∞ is the unique solution to*

$$u_\infty \in V^{1,q}(\Omega_\ell), \ E_{\Omega_\ell}(u_\infty) \le E_{\Omega_\ell}(v) \ \forall v \in V^{1,q}(\Omega_\ell). \tag{2.15}$$

(We suppose u_∞ extended to Ω_ℓ being taken constant in X_1).

Proof. First notice that $u_\infty \in V^{1,q}(\Omega_\ell)$. Indeed one has since u_∞ is independent of X_1

$$⨍_{\ell\omega_1} \nabla_{X_1} u_\infty \, dX_1 = 0.$$

Next let $u \in V^{1,q}(\Omega_\ell)$. Set

$$v(X_2) = ⨍_{\ell\omega_1} u(., X_2) \, dX_1.$$

It is easy to see that $v \in W_0^{1,q}(\omega_2)$. Thus

$$E_{\omega_2}(u_\infty) \leq E_{\omega_2}(v) = \int_{\omega_2} F\left(0, \nabla_{X_2} v\right) - fv \; dX_2.$$

Now using the fact that $u \in V^{1,q}(\Omega_\ell)$ one has

$$0 = \fint_{\ell\omega_1} \nabla_{X_1} u \; dX_1.$$

Also by differentiation under the integral, we get

$$\nabla_{X_2} v = \fint_{\ell\omega_1} \nabla_{X_2} u \; dX_1.$$

Therefore we have by Jensen's inequality

$$\begin{aligned} E_{\omega_2}(u_\infty) &\leq \int_{\omega_2} \left\{ F\left(\frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} \nabla_{X_1} u dX_1, \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} \nabla_{X_2} u dX_1 \right) \right\} dX_2 \\ &\quad - \int_{\omega_2} f \left\{ \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} u dX_1 \right\} dX_2 \\ &\leq \int_{\omega_2} \frac{1}{|\ell\omega_1|} \int_{\ell\omega_1} \{F(\nabla u) - fu\} \, dX_1 dX_2 = \frac{E_{\Omega_\ell}(u)}{|\ell\omega_1|}. \end{aligned}$$

Clearly equality holds in the above inequality if $u = u_\infty$. Then the claim follows from the uniqueness of the minimizer.

□

One can then show:

Lemma 2. *One has for $\ell' > \ell$*

$$0 \leq u_\ell \leq u_{\ell'} \leq u_\infty. \tag{2.16}$$

Proof. Recall that we are assuming $f \geq 0$. The nonnegativity of all the functions above results from the corollary 3. Indeed for $f = 0$ one has for instance

$$\begin{aligned} E_{\Omega_\ell}(v) = \int_{\Omega_\ell} F(\nabla v) dx &\geq \int_{\Omega_\ell} -\Lambda' dx \\ &= \int_{\Omega_\ell} F(0) dx = E_{\Omega_\ell}(0) \;\forall v \in W_0^{1,q}(\Omega_\ell) \end{aligned}$$

i.e. 0 is a minimizer. The monotonicity property $\ell \to u_\ell$ follows from the theorem 2. To prove that u_ℓ (or $u_{\ell'}$) is less than u_∞ one proceeds as follows. One considers

$$A_\ell = \{x \in \Omega_\ell \mid u_\ell(x) > u_\infty(x)\}.$$

One has

$$E_{A_\ell}(u_\ell) \leq E_{A_\ell}(u_\infty) \qquad (E_{A_\ell}(\cdot) = \int_{A_\ell} F(\nabla \cdot) - f \cdot \; dx). \tag{2.17}$$

Indeed, if not, setting $v_\ell = u_\ell \wedge u_\infty$ one has $v_\ell \in W_0^{1,q}(\Omega_\ell)$ and

$$E_{\Omega_\ell}(v_\ell) = E_{A_\ell}(v_\ell) + E_{A_\ell^C}(v_\ell) = E_{A_\ell}(u_\infty) + E_{A_\ell^C}(u_\ell) < E_{\Omega_\ell}(u_\ell)$$

and a contradiction with the definition of u_ℓ (A_ℓ^C denotes the set $\Omega\backslash A_\ell$). Assuming then (2.17) and considering $w_\ell = u_\ell \vee u_\infty$ one has $w_\ell \in V^{1,q}(\Omega_\ell)$. Indeed if ν denotes the normal to $\partial(\ell\omega_1)$ this follows from

$$\begin{aligned}\fint_{\ell\omega_1} \nabla_{X_1} w_\ell(.,X_2)\ dX_1 &= \frac{1}{|\ell\omega_1|}\int_{\partial(\ell\omega_1)} w_\ell(.,X_2)\nu\ d\sigma(X_1)\\ &= \frac{1}{|\ell\omega_1|}\int_{\partial(\ell\omega_1)} u_\infty(.,X_2)\nu\ d\sigma(X_1)\\ &= \fint_{\ell\omega_1} \nabla_{X_1} u_\infty(.,X_2)\ dX_1 = 0,\end{aligned}$$

(since $u_\infty(X_1,X_2)$ is constant in X_1). Then

$$E_{\Omega_\ell}(w_\ell) = E_{A_\ell}(w_\ell) + E_{A_\ell^C}(w_\ell) = E_{A_\ell}(u_\ell) + E_{A_\ell^C}(u_\infty) \le E_{\Omega_\ell}(u_\infty)$$

and thus $w_\ell = u_\infty$ -i.e. $u_\ell \le u_\infty$.

□

Clearly it follows from the lemma 2 that there exists $\tilde u_\infty$ such that

$$u_\ell \le \tilde u_\infty \le u_\infty\ \forall \ell,\ u_\ell \to \tilde u_\infty \text{ a.e. in } \mathbb{R}^p \times \omega_2. \tag{2.18}$$

Lemma 3. *The function $\tilde u_\infty$ is independent of X_1.*

Proof. For $i = 1, \cdots, p$ we set

$$\tau_h^i v(x) = v(x - he_i),\ h > 0$$

where e_i denotes the i-th vector of the canonical basis of $\mathbb{R}^p$. We claim that

$$u_{\ell+h} \ge \tau_{h'}^i u_\ell \quad \text{for } 0 < h' \le \lambda h \tag{2.19}$$

$\lambda \le 1$ being so small that

$$\lambda e_i \in \omega_1. \tag{2.20}$$

Indeed if (2.20) holds we have for $X_1 - h'e_i \in \ell\omega_1$ and some $Y_1 \in \omega_1$

$$X_1 = \ell Y_1 + h'e_i = (\ell+h)\{\frac{\ell}{\ell+h}Y_1 + \frac{h}{\ell+h}\frac{h'}{h}e_i\} \in (\ell+h)\omega_1$$

(since $Y_1, \frac{h'}{h}e_i \in \omega_1$ and ω_1 is convex containing 0). Thus the support of $\tau_{h'}^i u_\ell$ is contained in $\Omega_{\ell+h}$. One defines

$$\mathcal{A} = \{x \in \Omega_{\ell+h} \mid \tau_{h'}^i u_\ell(x) > u_{\ell+h}(x)\}.$$

One has $E_{\mathcal{A}}(u_{\ell+h}) \le E_{\mathcal{A}}(\tau_{h'}^i u_\ell)$. Indeed if not

$$w_\ell = u_{\ell+h} \vee \tau_{h'}^i u_\ell = u_{\ell+h} + (\tau_{h'}^i u_\ell - u_{\ell+h})^+ \in W_0^{1,q}(\Omega_{\ell+h})$$

and

$$\begin{aligned}E_{\Omega_{\ell+h}}(w_\ell) &= E_{\mathcal{A}}(\tau_{h'}^i u_\ell) + E_{\mathcal{A}^C}(u_{\ell+h})\\ &< E_{\mathcal{A}}(u_{\ell+h}) + E_{\mathcal{A}^C}(u_{\ell+h}) = E_{\Omega_{\ell+h}}(u_{\ell+h})\end{aligned}$$

and a contradiction with the definition of $u_{\ell+h}$. Let us set

$$\mathcal{A}' = \{x \in \Omega_\ell \mid \tau_{-h'}^i(u_{\ell+h})(x) < u_\ell(x)\}.$$

$$\begin{aligned} x \in \mathcal{A}' &\Leftrightarrow x + h'e_i \in \Omega_{\ell+h} \text{ and } u_{\ell+h}(x+h'e_i) < u_\ell(x) \\ &\Leftrightarrow y = x + h'e_i \in \Omega_{\ell+h} \text{ and } u_{\ell+h}(y) < \tau^i_{h'} u_\ell(y) \\ &\Leftrightarrow x + h'e_i \in \mathcal{A}, \end{aligned}$$

i.e. one has

$$\mathcal{A}' = \mathcal{A} - h'e_i. \tag{2.21}$$

Consider next

$$v_\ell = u_\ell \wedge \tau^i_{-h'}(u_{\ell+h}) = u_\ell - (u_\ell - \tau^i_{-h'}(u_{\ell+h}))^+.$$

Clearly $v_\ell \in W_0^{1,q}(\Omega_\ell)$ (recall that $\tau^i_{-h'}(u_{\ell+h}) \geq 0$). Then

$$E_{\Omega_\ell}(v_\ell) = E_{\mathcal{A}'}(\tau^i_{-h'}(u_{\ell+h})) + E_{\mathcal{A}'^C}(u_\ell).$$

Using (2.21) one has

$$\begin{aligned} E_{\mathcal{A}'}(\tau^i_{-h'}(u_{\ell+h})) &= \int_{\mathcal{A}'} \{F(\nabla u_{\ell+h}(x+h'e_i)) - f(X_2)u_{\ell+h}(x+h'e_i)\}dx \\ &= \int_{\mathcal{A}} \{F(\nabla u_{\ell+h}(y)) - f(X_2)u_{\ell+h}(y)\}dy = E_{\mathcal{A}}(u_{\ell+h}) \end{aligned}$$

and

$$\begin{aligned} E_{\mathcal{A}}(\tau^i_{h'}(u_\ell)) &= \int_{\mathcal{A}} \{F(\nabla u_\ell(x-h'e_i)) - f(X_2)u_\ell(x-h'e_i)\}\ dx \\ &= \int_{\mathcal{A}'} \{F(\nabla u_\ell(y)) - f(X_2)u_\ell(y)\}dy = E_{\mathcal{A}'}(u_\ell). \end{aligned}$$

Thus we obtain from above – recall that $E_{\mathcal{A}}(u_{\ell+h}) \leq E_{\mathcal{A}}(\tau^i_{h'}u_\ell)$

$$\begin{aligned} E_{\Omega_\ell}(v_\ell) =& E_{\mathcal{A}}(u_{\ell+h}) + E_{\mathcal{A}'^C}(u_\ell) \leq E_{\mathcal{A}}(\tau^i_{h'}u_\ell) + E_{\mathcal{A}'^C}(u_\ell) \\ &= E_{\mathcal{A}'}(u_\ell) + E_{\mathcal{A}'^C}(u_\ell) = E_{\Omega_\ell}(u_\ell). \end{aligned}$$

This is only possible if $v_\ell = u_\ell$ i.e. $(u_\ell - \tau^i_{-h'}(u_{\ell+h}))^+ = 0$ which is also

$$\tau^i_{h'}(u_\ell) \leq u_{\ell+h}.$$

Similary one would get

$$\tau^i_{-h'}(u_\ell) \leq u_{\ell+h}.$$

Thus, passing to the limit in ℓ in the inequalities above one derives

$$\tilde{u}_\infty(x - h'e_i) \leq \tilde{u}_\infty(x),\ \tilde{u}_\infty(x + h'e_i) \leq \tilde{u}_\infty(x) \ \text{ a.e. } x,$$

which implies

$$\tilde{u}_\infty(x) \leq \tilde{u}_\infty(x - h'e_i) \leq \tilde{u}_\infty(x) \ \text{ a.e. } x,\ \forall i = 1, \cdots, p,\ \forall h' \text{ small}.$$

This completes the proof of the lemma.

□

Lemma 4. *The function $\tilde{u}_\infty$ belongs to $W_0^{1,q}(\omega_2)$.*

Proof. Let $\rho = \rho_{\ell_0}$ be the function defined in (2.2). Since $(1-\rho)u_\ell \in W_0^{1,q}(\Omega_\ell)$ one derives from (2.8)

$$\int_{\Omega_\ell} \{F(\nabla u_\ell) - f u_\ell\} dx \leq \int_{\Omega_\ell} \{F(\nabla((1-\rho)u_\ell)) - f(1-\rho)u_\ell\} dx.$$

Since $(1-\rho) = 1$ outside of Ω_{ℓ_0+1} and 0 on Ω_{ℓ_0} we get

$$\begin{aligned}
&\int_{\Omega_{\ell_0+1}} \{F(\nabla u_\ell) - f u_\ell\} dx \\
&\qquad \leq \int_{\Omega_{\ell_0}} F(0) dx + \int_{D_{\ell_0}} \{F(\nabla((1-\rho)u_\ell)) - f(1-\rho)u_\ell\}\ dx \\
&\qquad \leq \int_{D_{\ell_0}} \{F(\nabla((1-\rho)u_\ell)) - f(1-\rho)u_\ell\} dx
\end{aligned}$$

(with $D_{\ell_0} = \Omega_{\ell_0+1} \backslash \Omega_{\ell_0}$) and thus

$$\begin{aligned}
\int_{\Omega_{\ell_0+1}} F(\nabla u_\ell) dx &\leq \int_{D_{\ell_0}} F(\nabla((1-\rho)u_\ell)) dx + \int_{\Omega_{\ell_0+1}} |f||u_\ell| dx \\
&\leq \int_{D_{\ell_0}} F(\nabla((1-\rho)u_\ell)) dx + |f|_{q',\Omega_{\ell_0+1}} |u_\ell|_{q,\Omega_{\ell_0+1}} \\
&\leq \int_{D_{\ell_0}} F(\nabla((1-\rho)u_\ell)) dx + C|f|_{q',\Omega_{\ell_0+1}} \big||\nabla u_\ell|\big|_{q,\Omega_{\ell_0+1}},
\end{aligned}$$

by the Poincaré inequality. Using the assumptions on F and Young's inequality it comes

$$\begin{aligned}
\lambda \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx - \Lambda'|\Omega_{\ell_0+1}| &\leq \Lambda \int_{D_{\ell_0}} |\nabla((1-\rho)u_\ell)|^q\ dx \\
&\quad + \epsilon \big||\nabla u_\ell|\big|^q_{q,\Omega_{\ell_0+1}} + C(\epsilon)|f|^{q'}_{q',\Omega_{\ell_0+1}} + \Lambda'|\Omega_{\ell_0+1}|.
\end{aligned}$$

Choosing $\epsilon = \frac{\lambda}{2}$ we get

$$\begin{aligned}
&\frac{\lambda}{2} \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx \\
&\quad \leq \Lambda \int_{D_{\ell_0}} |-u_\ell \nabla\rho + (1-\rho)\nabla u_\ell|^q\ dx + C(\lambda)|f|^{q'}_{q',\Omega_{\ell_0+1}} + 2\Lambda'|\Omega_{\ell_0+1}| \\
&\quad \leq C \int_{D_{\ell_0}} |\nabla u_\ell|^q\ dx + C(\lambda) \int_{(\ell_0+1)\omega_1} \int_{\omega_2} |f|^{q'}\ dX_2 dX_1 + 2\Lambda'|\Omega_{\ell_0+1}|
\end{aligned}$$

(we have used here the Poincaré inequality again). Thus we derive

$$\begin{aligned}
&\frac{\lambda}{2} \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx \\
&\qquad \leq C \int_{D_{\ell_0}} |\nabla u_\ell|^q\ dx + (\ell_0+1)^p |\omega_1| \{C(\lambda)|f|^{q'}_{q',\omega_2} + 2\Lambda'|\omega_2|\}.
\end{aligned}$$

If we still denote by C the maximum $\frac{2C}{\lambda} \vee \{\frac{2}{\lambda}|\omega_1|(C(\lambda)|f|^{q'}_{q',\omega_2} + 2\Lambda'|\omega_2|)\}$ we obtain

$$\int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q \, dx \leq C \int_{D_{\ell_0}} |\nabla u_\ell|^q \, dx + C(\ell_0+1)^p$$

where C is independent of ℓ_0 and ℓ. Thus it comes

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \leq C \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q \, dx - C \int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx + C(\ell_0+1)^p$$

and then

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \leq \gamma \int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q \, dx + \gamma(\ell_0+1)^p$$

with $\gamma = \frac{C}{C+1} < 1$. Iterating this inequality $[\ell - \ell_0]$ times where [ ] denotes the integer part of a number leads to

$$\begin{aligned}\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx &\leq \gamma\{\gamma \int_{\Omega_{\ell_0+2}} |\nabla u_\ell|^q \, dx + \gamma(\ell_0+2)^p\} + \gamma(\ell_0+1)^p \\ &\leq \gamma^2 \int_{\Omega_{\ell_0+2}} |\nabla u_\ell|^q \, dx + \gamma^2(\ell_0+2)^p + \gamma(\ell_0+1)^p \\ &\leq \gamma^{[\ell-\ell_0]} \int_{\Omega_{\ell_0+[\ell-\ell_0]}} |\nabla u_\ell|^q \, dx + \sum_{k=1}^{[\ell-\ell_0]} \gamma^k(\ell_0+k)^p.\end{aligned}$$

Using the fact that

$$\ell - \ell_0 - 1 < [\ell - \ell_0] \leq \ell - \ell_0$$

we derive

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \leq \frac{1}{\gamma^{\ell_0+1}} \, \gamma^\ell \int_{\Omega_\ell} |\nabla u_\ell|^q \, dx + \sum_{k=1}^{\infty} \gamma^k(\ell_0+k)^p. \tag{2.22}$$

Since

$$\lim_{k\to+\infty} \frac{\gamma^{k+1}(\ell_0+k+1)^p}{\gamma^k(\ell_0+k)^p} = \gamma < 1,$$

the series above converges and has for sum some real number that we will denote by $\Sigma(\ell_0)$. We know on the other hand taking $v = 0$ in (2.8) that

$$\int_{\Omega_\ell} F(\nabla u_\ell) \, dx - \int_{\Omega_\ell} f u_\ell \, dx \leq \int_{\Omega_\ell} F(0) \, dx$$

i.e.

$$\begin{aligned}\int_{\Omega_\ell} F(\nabla u_\ell) \, dx - \int_{\Omega_\ell} F(0) \, dx &\leq \int_{\Omega_\ell} f u_\ell dx \\ &\leq |f|_{q',\Omega_\ell} |u_\ell|_{q,\Omega_\ell} \leq C|f|_{q',\Omega_\ell} \big||\nabla u_\ell|\big|_{q,\Omega_\ell}\end{aligned}$$

by Hölder's inequality. Thus

$$\lambda \int_{\Omega_\ell} |\nabla u_\ell|^q \, dx \leq C|f|_{q',\Omega_\ell} \big||\nabla u_\ell|\big|_{q,\Omega_\ell}.$$

This leads to

$$\int_{\Omega_\ell} |\nabla u_\ell|^q \, dx \leq C \int_{\ell\omega_1} \int_{\omega_2} |f|^{q'} \, dX_2 dX_1 \leq C\ell^p.$$

Thus, going back to (2.22) we obtain

$$\int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx \le \frac{1}{\gamma^{\ell_0+1}} \, C\gamma^\ell \ell^p + \Sigma(\ell_0) \le K(\ell_0) \ \forall \ell \tag{2.23}$$

where $K(\ell_0)$ is a constant independent of ℓ. It follows that u_ℓ is bounded in $W^{1,q}_{lat}(\Omega_{\ell_0})$ and thus up to a subsequence one has that there exists $\hat{u}_\infty \in W^{1,q}_{lat}(\Omega_{\ell_0})$ such that

$$u_\ell \rightharpoonup \hat{u}_\infty \text{ in } W^{1,q}_{lat}(\Omega_{\ell_0}), \ u_\ell \to \hat{u}_\infty \text{ in } L^q(\Omega_{\ell_0}).$$

Since by the dominated convergence theorem one has

$$u_\ell \to \tilde{u}_\infty \text{ in } L^q(\Omega_{\ell_0})$$

one obtains

$$\hat{u}_\infty = \tilde{u}_\infty \in W^{1,q}_{lat}(\Omega_{\ell_0})$$

i.e. $\tilde{u}_\infty \in W^{1,q}_0(\omega_2)$. This completes the proof of the lemma.

□

Lemma 5. *One has*

$$\tilde{u}_\infty = u_\infty. \tag{2.24}$$

Proof. First notice that from (2.22), (2.23) one has

$$\begin{aligned} \int_{\Omega_{\ell_0}} |\nabla u_\ell|^q \, dx &\le 1 + \ell_0^p \sum_{k=1}^{+\infty} \gamma^k (1+k)^p \\ &= 1 + C\ell_0^p, \end{aligned} \tag{2.25}$$

for ℓ large enough i.e. $\ell \ge \ell(\ell_0)$, $\ell_0 \ge 1$. If $\rho = \rho_{\ell_0}$ is the function defined in (2.2) one has

$$(1-\rho)u_\ell + \rho u_\infty \in W^{1,q}_0(\Omega_\ell)$$

and thus from (2.8)

$$\begin{aligned} &\int_{\Omega_\ell} \{F(\nabla u_\ell) - f u_\ell\} \, dx \\ &\quad \le \int_{\Omega_\ell} \Big\{ F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - f\{(1-\rho)u_\ell + \rho u_\infty\} \Big\} \, dx. \end{aligned}$$

Since $(1-\rho)u_\ell + \rho u_\infty = u_\ell$ on $\Omega_\ell \backslash \Omega_{\ell_0+1}$ and is equal to u_∞ on Ω_{ℓ_0} we get

$$\begin{aligned} \int_{\Omega_{\ell_0+1}} \{F(\nabla u_\ell) - f u_\ell\} \, dx &\le \int_{\Omega_{\ell_0}} \{F(\nabla u_\infty) - f u_\infty\} \, dx \\ &+ \int_{D_{\ell_0}} \Big\{ F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - f\{(1-\rho)u_\ell + \rho u_\infty\} \Big\} \, dx. \end{aligned} \tag{2.26}$$

Let us denote by I the last integral above. One has

$$I = \int_{D_{\ell_0}} \Big\{F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty)$$

$$+ F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty) - f\{(1-\rho)u_\ell + \rho u_\infty\}\Big\}\ dx$$

$$\leq \int_{D_{\ell_0}} \Big\{F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) - F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty)\Big\}\ dx$$

$$+ \int_{D_{\ell_0}} (1-\rho)\{F(\nabla u_\ell) - fu_\ell\}\ dx + \rho\{F(\nabla u_\infty) - fu_\infty\}\ dx$$

due to the convexity of F. It follows from (2.26) that

$$\begin{aligned} \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\ell) - fu_\ell\}dx &\leq \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - fu_\infty\}dx \\ + \int_{D_{\ell_0}} \Big\{F(\nabla\{(1-\rho)u_\ell + \rho u_\infty\}) &- F((1-\rho)\nabla u_\ell + \rho\nabla u_\infty)\Big\}dx. \end{aligned} \tag{2.27}$$

Let J be the last integral above. Since

$$\nabla\{(1-\rho)u_\ell + \rho u_\infty\} - \{(1-\rho)\nabla u_\ell + \rho\nabla u_\infty\} = -(u_\ell - u_\infty)\nabla\rho$$

we deduce from (2.5)

$$J \leq C\int_{D_{\ell_0}} \Big(1 + |\nabla\{(1-\rho)u_\ell + \rho u_\infty\}|^{q-1}$$

$$+ |\{(1-\rho)\nabla u_\ell + \rho\nabla u_\infty\}|^{q-1}\Big)|u_\ell - u_\infty|dx$$

$$\leq C\int_{D_{\ell_0}} \Big(1 + |\nabla u_\ell|^{q-1} + |u_\ell|^{q-1} + |\nabla u_\infty|^{q-1} + |u_\infty|^{q-1}\Big)|u_\ell - u_\infty|dx$$

(we used the inequality $(a+b)^{q-1} \leq C(a^{q-1}+b^{q-1})$ valid for $q \geq 1,\ a, b \geq 0$). Using Hölder's inequality we derive

$$J \leq C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q\ dx\Big)^{\frac{1}{q}}$$

$$\Big\{(\int_{D_{\ell_0}} 1\ dx)^{\frac{1}{q'}} + (\int_{D_{\ell_0}} |\nabla u_\ell|^q\ dx)^{\frac{1}{q'}} + (\int_{D_{\ell_0}} |u_\ell|^q\ dx)^{\frac{1}{q'}}$$

$$+ (\int_{D_{\ell_0}} |\nabla u_\infty|^q\ dx)^{\frac{1}{q'}} + (\int_{D_{\ell_0}} |u_\infty|^q\ dx)^{\frac{1}{q'}}\Big\}.$$

Due to the Poincaré inequality it comes

$$J \leq C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q\ dx\Big)^{\frac{1}{q}}$$

$$\Big\{|D_{\ell_0}|^{1-\frac{1}{q}} + (\int_{\Omega_{\ell_0+1}} |\nabla u_\ell|^q\ dx)^{1-\frac{1}{q}} + (\int_{\Omega_{\ell_0+1}} |\nabla u_\infty|^q\ dx)^{1-\frac{1}{q}}\Big\}.$$

Using (2.25) we deduce that for ℓ large enough

$$J = J_\ell \le C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q \, dx\Big)^{\frac{1}{q}} (\ell_0+1)^{p(1-\frac{1}{q})}.$$

Thus, for large ℓ, (2.27) can be written as

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\ell) - f u_\ell\} \, dx \le \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\} \, dx \\ + C\Big(\int_{D_{\ell_0}} |u_\ell - u_\infty|^q \, dx\Big)^{\frac{1}{q}} (\ell_0+1)^{p(1-\frac{1}{q})}.\end{aligned} \tag{2.28}$$

Passing to the lim inf and using the weak lower semicontinuity of the left hand side (recall that F is convex) we get

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla \tilde u_\infty) - f \tilde u_\infty\} \, dx \le \int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\} \, dx \\ + C\Big(\int_{D_{\ell_0}} |\tilde u_\infty - u_\infty|^q \, dx\Big)^{\frac{1}{q}} (\ell_0+1)^{p(1-\frac{1}{q})}.\end{aligned}$$

Considering each of the terms above one has

$$\begin{aligned}\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla \tilde u_\infty) - f \tilde u_\infty\} dx &= \int_{(\ell_0+1)\omega_1} \int_{\omega_2} \rho\{F(\nabla \tilde u_\infty) - f \tilde u_\infty\} dx \\ &= E_{\omega_2}(\tilde u_\infty) \int_{(\ell_0+1)\omega_1} \rho dX_1.\end{aligned}$$

Similarly

$$\int_{\Omega_{\ell_0+1}} \rho\{F(\nabla u_\infty) - f u_\infty\} \, dx = E_{\omega_2}(u_\infty) \int_{(\ell_0+1)\omega_1} \rho dX_1.$$

Finally

$$\begin{aligned}\Big(\int_{D_{\ell_0}} |\tilde u_\infty - u_\infty|^q \, dx\Big)^{\frac{1}{q}} &= \Big(\int_{(\ell_0+1)\omega_1 \setminus \ell_0\omega_1} \int_{\omega_2} |\tilde u_\infty - u_\infty\}|^q \, dx\Big)^{\frac{1}{q}} \\ &= |\tilde u_\infty - u_\infty|_{q,\omega_2} |(\ell_0+1)\omega_1 \setminus \ell_0\omega_1|^{\frac{1}{q}} \\ &= |\tilde u_\infty - u_\infty|_{q,\omega_2} (|(\ell_0+1)\omega_1| - |\ell_0\omega_1|)^{\frac{1}{q}} \\ &\le C\ell_0^{(p-1)\frac{1}{q}}.\end{aligned}$$

where C is independent of ℓ_0. Thus we have obtained

$$\begin{aligned}E_{\omega_2}(\tilde u_\infty) &\le E_{\omega_2}(u_\infty) + C\ell_0^{(p-1)\frac{1}{q}} \ell_0^{p(1-\frac{1}{q})} / \int_{\Omega_{\ell_0+1}} \rho \, dX_1 \\ &\le E_{\omega_2}(u_\infty) + C\frac{\ell_0^{(p-1)\frac{1}{q}} \ell_0^{p(1-\frac{1}{q})}}{\ell_0^p}\end{aligned}$$

for some constant C, i.e.

$$E_{\omega_2}(\tilde u_\infty) \le E_{\omega_2}(u_\infty) + C\ell_0^{-\frac{1}{q}}.$$

Letting $\ell_0 \to +\infty$ we obtain $E_{\omega_2}(\tilde u_\infty) \le E_{\omega_2}(u_\infty)$. By definition of u_∞ this implies that $\tilde u_\infty = u_\infty$ and the result follows.

□

Remark 5. We have now proved the theorem 5. Note that the convergence (2.12) takes place pointwise and also in $L^q(\Omega_{\ell_0})$-strong and $W^{1,q}_{lat}(\Omega_{\ell_0})$-weak for any ℓ_0.

Remark 6. The function

$$F(\xi) = \frac{1}{q}|\xi|^q$$

satisfies the assumptions (2.3), (2.4). The solution of the minimisation problem (2.8) is also weak solution of the q-Laplace Dirichlet problem

$$\begin{cases} -\nabla \cdot (|\nabla u_\ell|^{q-2}\nabla u_\ell) = f \text{ in } \Omega_\ell, \\ u_\ell = 0 \text{ on } \partial\Omega_\ell. \end{cases}$$

One recovers the results of [12] i.e. the convergence of u_ℓ toward the weak solution u_∞ to

$$\begin{cases} -\nabla_{X_2} \cdot (|\nabla_{X_2} u_\infty|^{q-2}\nabla_{X_2} u_\infty) = f \text{ in } \omega_2, \\ u_\infty = 0 \text{ on } \partial\omega_2, \end{cases}$$

with a clear meaning for $\nabla_{X_2}\cdot$ i.e. the divergence in X_2.

To conclude this section we would like to consider the asymptotic behavior of

$$\frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|}$$

when $\ell \to \infty$.

In particular we would like to prove (Cf. [8]):

Theorem 6. *(***Convergence of the energy***) One has for some constant $C > 0$ and sufficiently large ℓ,*

$$E_{\omega_2}(u_\infty) \le \frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|} \le E_{\omega_2}(u_\infty) + \frac{C}{\ell}$$

where u_ℓ, u_∞ are the solutions to (2.8) *and* (2.11) *respectively.*

Proof. Since $u_\ell \in W^{1,q}_0(\Omega_\ell) \subset V^{1,q}(\Omega_\ell)$ the left hand side inequality is an immediate consequence of (2.15) which can be written for $v = u_\ell$

$$|\ell\omega_1| E_{\omega_2}(u_\infty) = E_{\Omega_\ell}(u_\infty) \le E_{\Omega_\ell}(u_\ell).$$

To prove the right hand side inequality, one considers $\rho = \rho_{\ell-1}(X_1)$, as defined in (2.2). Since $\rho u_\infty \in W^{1,q}_0(\Omega_\ell)$ from (2.8), we have

$$E_{\Omega_\ell}(u_\ell) \le E_{\Omega_\ell}(\rho u_\infty).$$

Thus from (2.2), (2.4) we deduce for some constant C

$$\begin{aligned}
E_{\Omega_\ell}(\rho u_\infty) &= E_{\Omega_{\ell-1}}(u_\infty) + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} F\big(\nabla(\rho u_\infty)\big) - f u_\infty \rho dx \\
&\le E_{\Omega_\ell}(u_\infty) + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} F\big(\nabla(\rho u_\infty)\big) - F\big(\nabla u_\infty\big) - f u_\infty(\rho - 1) dx \\
&\le |\ell\omega_1| E_{\omega_2}(u_\infty) + \int_{\Omega_\ell \setminus \Omega_{\ell-1}} 2\Lambda' + \Lambda|\nabla(\rho u_\infty)|^q + \Lambda|\nabla u_\infty|^q + |f||u_\infty| dx \\
&\le |\ell\omega_1| E_{\omega_2}(u_\infty) + C\int_{\Omega_\ell \setminus \Omega_{\ell-1}} 1 + |\nabla u_\infty|^q + |u_\infty|^q + |f||u_\infty| dx \\
&\le |\ell\omega_1| E_{\omega_2}(u_\infty) + C\{|\ell\omega_1| - |(\ell-1)\omega_1|\}\int_{\omega_2} 1 + |\nabla u_\infty|^q + |u_\infty|^q + |f||u_\infty| dx.
\end{aligned}$$

Dividing by $|\ell\omega_1|$ the result follows, i.e. one has for some other constant C

$$\frac{E_{\Omega_\ell}(u_\ell)}{|\ell\omega_1|} \le E_{\omega_2}(u_\infty) + \frac{C}{\ell}.$$

This completes the proof of the theorem. □

References

[1] M. Chipot, *ℓ goes to plus infinity*. Birkhäuser Advanced Text, 2002.

[2] M. Chipot : ℓ goes to to plus infinity : an update, J. KSIAM, vol 18, 2, 107–127.

[3] M. Chipot, *Elements of Nonlinear Analysis*, Birkhäuser Advanced Text, 2000.

[4] M. Chipot, *Elliptic Equations: An Introductory Course*, Birkhäuser Advanced Text, 2009.

[5] M. Chipot, *Asymptotic Issues for Some Partial Differential Equations,* Imperial College Press, 2016.

[6] M. Chipot, S. Mardare, *Asymptotic behaviour of the Stokes problem in cylinders becoming unbounded in one direction,* J. Math. Pures Appl. 90, (2008), 133-159.

[7] M. Chipot, S. Mardare, *The Neumann problem in cylinders becoming unbounded in one direction.* J. Math. Pures App. (9) 104 (2015), no. 5, 921-941.

[8] M. Chipot, A. Mojsic and P. Roy *On some variational problems set on domains tending to infinity.* Discrete Contin. Dyn. Syst. Series A, 36 (7), (2016), 3603-3621.

[9] M. Chipot, A. Rougirel, *On the asymptotic behaviour of the eigenmodes for elliptic problems in domains becoming unbounded,* Trans. Amer. Math. Soc. 360 (2008), 3579-3602.

[10] M. Chipot, K. Yeressian, *Exponential rates of convergence by an iteration technique,* C. R. Acad. Sci. Paris, Ser. I 346, (2008), 21-26.

[11] M. Chipot, K. Yeressian, *On Some Variational Inequalities in Unbounded Domains,* Boll. del UMI, (9), V, (2012), 243-262.

[12] M. Chipot, Y. Xie, *Some issues on the p-Laplace equation in cylindrical domains,* Proceedings of the Steklov Institue of Mathematics, 261, (2008), 287-294.

[13] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Applied Mathematical Sciences Series # 78, Second Edition, Springer, 2008.

[14] R. Dautray, J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, Springer-Verlag, 1988.

[15] L.C. Evans, *Partial differential equations*. Graduate Studies in Mathematics 19, AMS, 1998.

Institute of Applied Mathematics, University of Zurich, Winterthurerstrasse 190, CH-8057 Zurich

E-mail address: m.m.chipot@math.uzh.ch

URL: http://user.math.uzh.ch/chipot/

JEPE Vol 1, 2015, p. 325-333

Orthogonal $\perp$

DECAY ESTIMATES FOR LINEARIZED UNSTEADY INCOMPRESSIBLE VISCOUS FLOWS AROUND ROTATING AND TRANSLATING BODIES

PAUL DEURING, STANISLAV KRAČMAR, ŠARKA NEČASOVÁ, AND PETER WITTWER

Abstract. We consider the time-dependent Oseen sytem with rotational terms. This system is a linearized model for the flow of a viscous incompressible fluid around a rigid body moving at a constant velocity and rotating with constant angular velocity. We present results on temporal and spatial decay of solutions to this system in the whole space. The spatial asymptotics we establish exhibit a wake.

1. Introduction

Consider the motion of a viscous incompressible fluid around a rigid body translating with constant velocity and rotating at constant angular velocity. Suppose the fluid flow is described with respect to a coordinate system in which the body is at rest. Then the flow in question is usually represented by a modified Navier-Stokes system which reads like this:

$$\partial_t v - \nu\,\Delta_z v + (v\cdot\nabla_z)v - (V+\omega\times z)\cdot\nabla_z v + \omega\times v + \nabla_z q = F, \quad \operatorname{div}_z v = 0 \tag{1.1}$$

in $(\mathbb{R}^3\backslash\overline{\mathfrak{D}})\times(0,T)$. Here $\mathfrak{D}\subset\mathbb{R}^3$ is a bounded domain representing the rigid body. The function v denotes the velocity field of the fluid, and the function q its pressure field. The vector V describes the constant translation of the body, and the vector ω its constant angular velocity. We suppose that V and ω are parallel. The function F stands for an exterior force exerted on the fluid, and the parameter $\nu\in(0,\infty)$ characterizes the viscosity of the fluid. By a suitable normalization and some changes of variables (see [16]), system (1.1) may be rewritten in the form

$$\partial_t u - \Delta_x u + \tau\,\partial_{x_1}u + \tau(u\cdot\nabla_x)u - (\varrho\,e_1\times x)\cdot\nabla_x u + \varrho\,e_1\times u + \nabla_x\sigma = f, \;\; \operatorname{div}_x u = 0, \tag{1.2}$$

where $\tau\in(0,\infty)$ is the Reynolds number and $\varrho\in\mathbb{R}\backslash\{0\}$ the Taylor number.

In recent years, many articles dealt with flows around a rotating body. As examples we mention [10] – [14], [17] – [18], [20] – [21]. In the present context, an article by Chen and Miyakava [1] is relevant. These authors proved existence of a global weak solution to (1.1) in the whole space $\mathbb{R}^n$ with $n=2$ and $n=3$, and derived algebraic decay rates (as $t\to\infty$) for the kinetic energy associated with this solution. They assumed $F=0$, $\nu=1$ but considered

2010 *Mathematics Subject Classification.* 35Q30, 76D05.

Key words and phrases. Whole space, viscous incompressible flow, rotating body, fundamental solution, Navier-Stokes system.

Received 31/12/2016, accepted 29/02/2016.

The research of Š. N. and S. K. was supported by Grant Agency of Czech Republic P201-16-03230S. Moreover research of Š.N. was supported by RVO 67985840.

nonzero initial data and admitted the case that V and ω are functions depending on time, and need not be parallel. We will show results related to those in [1], but pertaining to the Oseen system with rotational terms, that is, to the following system obtained by dropping the nonlinearity in (1.2),

$$\partial_t u - \Delta_x u + \tau\,\partial_{x_1} u - (\varrho\, e_1 \times x)\cdot \nabla_x u + \varrho\, e_1 \times u + \nabla_x \sigma = f, \quad \mathrm{div}_x u = 0. \tag{1.3}$$

Under the assumption that f does not depend on time and decays in an appropriate way, we will study the asymptotics of $U(x)-u(x,t)$ and $\nabla_x\big(U(x)-u(x,t)\big)$ with respect to both the space variable x and the time variable t, where u is the velocity part of a solution to (1.3) with initial data zero, and U the velocity part of a solution to the stationary variant of (1.3), that is,

$$-\Delta U + \tau\,\partial_1 U - (\varrho\, e_1 \times x)\cdot \nabla_x U + \varrho\, e_1 \times U + \nabla \Pi = f, \quad \mathrm{div}\, U = 0. \tag{1.4}$$

The decay bounds we obtain for $U(x) - u(x,t)$ exhibit a wake. In addition, they imply optimal rates of spatial decay for $u(x,t)$ when $|x| \to \infty$. These rates are uniform with respect to t. Our estimates of $U(x)-u(x,t)$ further yield that $u(\,\cdot\,,t)$ converges to U with respect to a weighted $W^{1,\infty}$-norm, which we will denote by $\|\ \|_{1,\infty,w,\epsilon}$. The rate of this convergence is $t^{-\epsilon}$, where ϵ may be arbitrarily chosen in $(0,\,1/2)$ but enters into the definition of $\|\ \|_{1,\infty,w,\epsilon}$. This convergence result means in particular that U is unconditionally asymptotically stable with respect to the norm $\|\ \|_{1,\infty,w,\epsilon}$. For more details on our results we refer to Theorem 2, Corollary 1 and the comments in Section 4.

2. Notations, definitions and auxiliary results

If $A \subset \mathbb{R}^3$, we write A^c for the complement $\mathbb{R}^3\backslash A$ of A. The symbol $|\ |$ denotes the Euclidean norm of $\mathbb{R}^3$ and also the length of a multiindex from $\mathbb{N}_0^3$, that is, $|\alpha| := \alpha_1+\alpha_2+\alpha_3$ for $\alpha \in \mathbb{N}_0^3$. The open ball centered at $x \in \mathbb{R}^3$ and with radius $r > 0$ is denoted by $B_r(x)$. If $x = 0$, we will write B_r instead of $B_r(0)$. Put $e_1 := (1,0,0)$. Let $x \times y$ denote the usual vector product of $x, y \in \mathbb{R}^3$.

The parameters $\tau \in (0,\infty)$ and $\varrho \in \mathbb{R}\backslash\{0\}$ will be kept fixed throughout. Put $s_\tau(x) := 1 + \tau\ (|x| - x_1)$ for $x \in \mathbb{R}^3$. Define the matrix $\Omega \in \mathbb{R}^{3\times 3}$ by

$$\Omega := \varrho \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

so that $\varrho\, e_1 \times x = \Omega \cdot x$ for $x \in \mathbb{R}^3$. By the symbol $\mathfrak{C}$, we denote constants only depending on τ or ω. We write $\mathfrak{C}(\gamma_1, ..., \gamma_n)$ for constants that additionally depend on parameters γ_1, ..., $\gamma_n \in \mathbb{R}$, for some $n \in \mathbb{N}$.

For $p \in [1,\infty)$ and for open sets $A \subset \mathbb{R}^3$, we write $W^{1,p}(A)$ for the usual Sobolev space of order 1 and exponent p. If $B \subset \mathbb{R}^3$ is open, define $W^{1,p}_{loc}(B)$ as the set of all functions $g : B \mapsto \mathbb{R}$ such that $g|U \in W^{1,p}(U)$ for any open bounded set $U \subset \mathbb{R}^3$ with $\overline{U} \subset B$. If V is a normed space whose norm is denoted by $\|\ \|_V$, and if $n \in \mathbb{N}$, we equip the product space V^n with a norm $\|\ \|_V^{(n)}$ defined by $\|v\|_V^{(n)} := \Big(\sum_{j=1}^n \|v_j\|_V^2\Big)^{1/2}$ for $v \in V^n$. But for simplicity, we will write $\|\ \|_V$ instead of $\|\ \|_V^{(n)}$.

Let K denote the usual fundamental solution to the heat equation,

$$K(x,t) = (4\pi t)^{-3/2} e^{-|x|^2/(4t)} \text{ for } x \in \mathbb{R}^3,\ t \in (0,\infty). \tag{2.1}$$

Recall that the Kummer function ${}_1F_1(1,\,\cdot\,,\,\cdot\,)$ is given by

$$ {}_1F_1(1,c,u) := \sum_{n=0}^{\infty}(\Gamma(c)/\Gamma(u+c))u^n \text{ for all } u \in \mathbb{R}, c \in (0,\infty), $$

where Γ denotes the usual Gamma function. We put

$$ H_{jk}(x) := x_j x_k |x|^{-2} \text{ for } x \in \mathbb{R}^3 \setminus \{0\}, $$
$$ \Lambda_{jk}(x,t) := K(x,t)(\delta_{jk} - H_{jk}(x) - {}_1F_1(1,5/2,|x|^2/(4t)))(\delta_{jk}/3 - H_{jk}(x)) $$

for $x \in \mathbb{R}^3 \setminus \{0\}$, $t \in (0,\infty)$, $j,k \in \{1,2,3\}$. In what follows, the letter Γ will always stand for a matrix-valued function defined by

$$ (\Gamma_{jk}(y,z,t))_{1 \le j,k \le 3} := (\Lambda_{rs}(y - \tau t e_1 - e^{-t\Omega} \cdot z, t))_{1 \le r,s \le 3} \cdot e^{-t\Omega} $$

for $y, z \in \mathbb{R}^3$, $t \in (0,\infty)$ with $y - \tau t e_1 - e^{-t\Omega} \cdot z \neq 0$.

This function is the velocity part of the fundamental solution to (1.3) introduced by Guenther, Thomann [22]. Our following lemma restates [3, Corollary 3.1].

Lemma 1. *The function Γ_{jk} may be extended continuously to a function belonging to $C^\infty\big(\mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)\big)$.*

We will use the ensuing technical lemmas:

Lemma 2. *(see* [4, Lemma 2.9]*) Let $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. Then*

$$ |e^{t\Omega} \cdot x| = |x|, \;\; (e^{t\Omega} \cdot x)_1 = x_1, \;\; e^{t\Omega} \cdot e_1 = e_1. $$

Lemma 3. *(see* [2, Lemma 4.8]*) For $x, y \in \mathbb{R}^3$ we have*

$$ s_\tau(x-y)^{-1} \le \mathfrak{C}\,(1+|y|) s_\tau(x)^{-1}. $$

Lemma 4. *(see* [9, Lemma 4.3]*) Let $\beta \in (1,\infty)$. Then*

$$ \int_{\partial B_r} s_\tau(x)^{-\beta} do_x \le \mathfrak{C}(\beta) r \text{ for } r \in (0,\infty). $$

Lemma 5. *(see* [4]*, Lemma 2.4) Let $S \in (0,\infty)$, $x \in B_S^c$. Then*

$$ |x| \ge \mathfrak{C}(S)\, s_\tau(x). $$

Lemma 6. *(see* [3, Lemma 3.2]*)*

$$ |\partial_y^\beta \Gamma_{jk}(y,z,t)| + |\partial_z^\beta \Gamma_{jk}(y,z,t)| \le \mathfrak{C}(|y - \tau t e_1 - e^{t\Omega} \cdot z|^2 + t)^{-3/2-|\beta|/2} \tag{2.2} $$

for $y, z \in \mathbb{R}^3$, $t \in (0,\infty)$, $\beta \in \mathbb{N}_0^3$ with $|\beta| \le 1$.

Lemma 7. *(see* [3, Theorem 3.1]*) Let $k \in \{0,1\}$, $R \in (0,\infty)$, $y, z \in B_R$ with $y \neq z$. Then*

$$ \int_0^\infty (|y - \tau t e_1 - e^{-t\Omega} \cdot z|^2 + t)^{-3/2-k/2} dt \le \mathfrak{C}(R)\, |y-z|^{-1-k}. \tag{2.3} $$

Due to the preceding lemma and by (2.2), we may define

$$ \mathfrak{Z}_{jk}(y,z,T) := \int_T^\infty \Gamma_{jk}(y,z,t) dt $$

for $T \in [0,\infty)$, $y, z \in \mathbb{R}^3$ with $y \neq z$, $1 \le j,k \le 3$. The function $\mathfrak{Z}(\,\cdot\,,\,\cdot\,,0)$ is the velocity part of the fundamental solution of (1.3) proposed by Guenther, Thomann [22].

Lemma 8. *Let $j, k \in \{1,2,3\}$, $T \in [0,\infty)$. Then $\mathfrak{Z}_{jk}(\,\cdot\,,\,\cdot\,,T) \in C^1\big((\mathbb{R}^3 \times \mathbb{R}^3) \setminus \{(x,x) : x \in \mathbb{R}^3\}\big)$, and*

$$(2.4)\ \partial_{y_n}\mathfrak{Z}_{jk}(y,z,T) = \int_T^\infty \partial_{y_n}\Gamma_{jk}(y,z,t)dt,\ \partial_{z_n}\mathfrak{Z}_{jk}(y,z,T) = \int_T^\infty \partial z_n\Gamma_{jk}(y,z,t)dt$$

for $y, z \in \mathbb{R}^3$ with $y \neq z$, $n \in \{1,2,3\}$. If $R \in (0,\infty)$, $y, z \in B_R$ with $y \neq z$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, we have

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z,T)| \leq \mathfrak{C}(R)\,|y-z|^{-1-|\alpha|}. \tag{2.5}$$

Proof. The proof of [4, Lemma 2.15] carries over to the present situation ($T \in [0,\infty)$ instead of $T = 0$). Note that (2.5) follows from (2.4) and (2.3). □

Theorem 1. *Let $S, \delta \in (0,\infty)$, $\nu \in (1,\infty)$, $T \in (0,\infty)$ and $0 \leq \epsilon < \nu - 1$, or $T = 0$ and $\epsilon = 0$. Then*

$$\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu} dt \leq \mathfrak{C}(S,\delta,\epsilon,\nu)T^{-\epsilon}(|y|s_\tau(y))^{-\nu+\epsilon+1/2} \tag{2.6}$$

for $y \in B^c_{(1+\delta)S}$, $z \in \overline{B_S}$, and

$$\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu} dt \leq \mathfrak{C}(S,\delta,\epsilon,\nu)T^{-\epsilon}(|z|s_\tau(z))^{-\nu+\epsilon+1/2} \tag{2.7}$$

for $z \in B^c_{(1+\delta)S}$, $y \in \overline{B_S}$. Moreover,

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z,T)| \leq \mathfrak{C}(s,\delta,\epsilon)\,T^{-\epsilon}(|y|s_\tau(y))^{-1-|\alpha|/2+\epsilon} \tag{2.8}$$

for $j, k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, $y \in B^c_{(1+\delta)S}$, $z \in \overline{B_S}$,

$$|\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)| + |\partial_z^\alpha \mathfrak{Z}_{jk}(y,z,T)| \leq \mathfrak{C}(s,\delta,\epsilon)\,T^{-\epsilon}(|z|s_\tau(z))^{-1-|\alpha|/2+\epsilon} \tag{2.9}$$

for $y \in \overline{B_S}$, $z \in B^c_{(1+\delta)S}$ and j, k, α as above.

Proof. In the case $T = 0$, $\epsilon = 0$, Theorem 1 restates [4, Theorem 2.19]. Now suppose that $T > 0$ and $0 \leq \epsilon < \nu - 1$. Then inequalities (2.6) and (2.7) may be reduced to the preceding reference. In fact, take y, z as in (2.6) or (2.7). Then

$$\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu} dt \leq \int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu+\epsilon}\, t^{-\epsilon} dt$$
$$\leq T^{-\epsilon}\int_T^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu+\epsilon} dt \leq T^{-\epsilon}\int_0^\infty (|y - \tau e_1 - e^{-t\Omega}\cdot z|^2 + t)^{-\nu+\epsilon} dt.$$

Since $-\nu + \epsilon < -1$, we may now use [4, Theorem 2.19] with ν replaced by $\nu - \epsilon$, obtaining (2.6) and (2.7). The estimates in (2.8) and (2.9) follow from (2.2), (2.4), (2.6) and (2.7). □

3. Volume potentials

We will study volume potentials involving the kernel $\mathfrak{Z}(\,\cdot\,,T)$.

Lemma 9. *Let $p \in (1,\infty)$, $q \in (1,2)$, $f \in L^p_{\mathrm{loc}}(\mathbb{R}^3)^3$ with $f|B^c_S \in L^q(B^c_S)^3$ for some $S \in (0,\infty)$. Then, for $j, k \in \{1,2,3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, we have*

$$\int_{\mathbb{R}^3}\int_0^\infty |\partial_y^\alpha \Gamma(y,z,t)|\,dt\,|f_k(z)|dz < \infty \text{ for a.e. } y \in \mathbb{R}^3. \tag{3.1}$$

Let $T \in [0,\infty)$. We define $\mathfrak{R}(f)(\,\cdot\,,T) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ by putting

$$\mathfrak{R}_j(f)(y,T) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} \int_T^\infty \Gamma_{jk}(y,z,t)dt\, f_k(z)dz = \int_{\mathbb{R}^3} \sum_{k=1}^{3} \mathfrak{Z}_{jk}(y,z,T)\, f_k(z)dz$$

for $y \in \mathbb{R}^3$ such that (3.1) holds; otherwise we set $\mathfrak{R}_j(t)(y,T) := 0$ $(1 \le j \le 3)$. Then $\mathfrak{R}(f)(\,\cdot\,,T) \in W^{1,1}_{\text{loc}}(\mathbb{R}^3)^3$ and

$$\partial_l \mathfrak{R}_j(f)(y,T) = \int_{\mathbb{R}^3} \sum_{k=1}^{3} \partial_{y_l} \mathfrak{Z}_{jk}(y,z,T)\, f_k(z)dz \tag{3.2}$$

for $j,l \in \{1,2,3\}$ and for a.e. $y \in \mathbb{R}^3$. Moreover, if $f \in L^1(\mathbb{R}^3)^3$, $T > 0$, we have

$$|\partial^\alpha \mathfrak{R}(f)(y,T)| \le \mathfrak{C}\, T^{-\frac{1}{2}-\frac{|\alpha|}{2}} \|f\|_1 \quad \text{for } \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1,\ y \in \mathbb{R}^3. \tag{3.3}$$

Proof. In view of (2.8) and (2.9) with $\epsilon = 0$, and due to Lemma 8, all the statements of Lemma 9 except (3.3) may be proved in exactly the same way, without any modification, as analogous statements in [4, Lemma 3.1]. As for (3.3), we use (2.2) to obtain for $y \in \mathbb{R}^3$, $1 \le j,k \le 3$ that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \int_T^\infty |\partial_y^\alpha \Gamma_{jk}(y,z,t)| dt\, |f(z)|\, dz \\
&\le \mathfrak{C} \int_{\mathbb{R}^3} \int_T^\infty (|y - \tau t e_1 - e^{-t\Omega} \cdot z|^2 + t)^{-3/2-|\alpha|/2} dt\, |f(z)|\, dz \\
&\le \mathfrak{C} \int_{\mathbb{R}^3} \int_T^\infty t^{-3/2-|\alpha|/2} dt\, |f(z)|\, dz \le \mathfrak{C}\, T^{-1/2-|\alpha|/2} \|f\|_1 .
\end{aligned}$$

Inequality (3.3) now follows with (3.2) and (2.4). □

Theorem 2. *Let $T \in (0,\infty)$, S, S_1, $\gamma \in (0,\infty)$ with $S_1 < S$, $p \in (1,\infty)$, $A \in [2,\infty)$, $B \in \mathbb{R}$, $0 \le \epsilon < 1/2 + |\alpha|/2$, $f : \mathbb{R}^3 \mapsto \mathbb{R}^3$ measurable with*

$$f|B_{S_1} \in L^p(B_{S_1})^3, \quad |f(z)| \le \gamma |z|^{-A} s_\tau(z)^{-B} \text{ for } z \in B_{S_1}^c, \quad A + \min\{1,B\} \ge 3.$$

Let $i,j \in \{1,\,2,\,3\}$, $y \in B_S^c$. Then

$$|\mathfrak{R}_j(f)(y,T)| \le \mathfrak{C}(S,S_1,A,B,\epsilon) T^{-\epsilon} (\|f|B_{S_1}\|_1 + \gamma) \big(|y|\, s_\tau(y)\big)^{-1+\epsilon} l_{A,B}(y), \tag{3.4}$$

$$|\partial_{y_i} \mathfrak{R}_j(f)(y,T)| \le \mathfrak{C}(S,S_1,A,B,\epsilon) T^{-\epsilon} (\|f|B_{S_1}\|_1 + \gamma) \big(|y|\, s_\tau(y)\big)^{-3/2+\epsilon} l_{A,B}(y)\, s_\tau(y)^{\max(0,\,7/2-A-B-2\epsilon)}, \tag{3.5}$$

where

$$l_{A,B}(y) = \begin{cases} 1 & \text{if } A + \min\{1,B\} > 3 \\ \max(1, \ln|y|) & \text{if } A + \min\{1,B\} = 3 \end{cases}$$

Proof. We modify the proof of [4, Theorem 3.1]. Since $A \ge 2$, we have $f|B_{S_1}^c \in L^q(B_{S_1}^c)^3$ for any $q \in (3/2,\infty)$. But $f|B_{S_1} \in L^p(B_{S_1})^3$, so we get, say, $f \in L^{\min\{p,2\}}_{loc}(\mathbb{R}^3)^3$. Therefore f satisfies the assumptions of Lemma 9, hence $\mathfrak{R}(f)(\,\cdot\,,T)$ is well defined, belongs to $W^{1,1}_{loc}(\mathbb{R}^3)^3$ and verifies (3.2).

By (2.8), we find for $k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$ that

$$\int_{B_{S_1}} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)|\,|f(z)|\,dz \le \mathfrak{C}(S,S_1,\epsilon)T^{-\epsilon}\big(\,|y|\,s_\tau(y)\,\big)^{-1-|\alpha|/2+\epsilon}\|f|B_{S_1}\|_1. \tag{3.6}$$

We further get with (2.4), (2.2), a change of variables and Lemma 2 that

$$\begin{aligned}
\mathfrak{A}_\alpha &:= \int_{B_{S_1}^c} |\partial_y^\alpha \mathfrak{Z}_{jk}(y,z,T)|\,|f(z)|\,dz \\
&\le \mathfrak{C}\gamma \int_T^\infty \int_{B_{S_1}^c} (|y-\tau t e_1 - e^{-t\Omega}\cdot z|^2+t)^{-3/2-|\alpha|/2}\,|z|^{-A}\,s_\tau(z)^{-B}\,dz\,dt \\
&= \mathfrak{C}\gamma \int_T^\infty \int_{B_{S_1}^c} (|y-\tau t e_1 - x|^2+t)^{-3/2-|\alpha|/2}\,|x|^{-A}\,s_\tau(e^{t\Omega}\cdot x)^{-B}\,dx\,dt \\
&= \mathfrak{C}\gamma T^{-\epsilon} \int_{B_{S_1}^c} \int_T^\infty (|y-\tau t e_1 - x|^2+t)^{-3/2-|\alpha|/2+\epsilon}\,dt\,|x|^{-A}\,s_\tau(x)^{-B}\,dx.
\end{aligned} \tag{3.7}$$

The preceding integral over $B_{S_1}^c$ is split into a sum of integrals over $B_{S_1}^c \cap B_{S/2}(y)$ and $B_{S_1}^c \backslash B_{S/2}(y)$, respectively. In order to estimate the integral over $B_{S_1}^c \cap B_{S/2}(y)$, we observe that for $x \in \mathbb{R}^3$, the term $(|y-\tau t e_1 - x|^2+t)^{-3/2-|\alpha|/2+\epsilon}$ is bounded by $(|y-\tau t e_1 - x|^2+t)^{-2}$ if $|y-\tau t e_1 - x|^2 + t \le 1$. Else it may be bounded by $\min\{1,\, t^{-3/2-|\alpha|/2+\epsilon}\}$. Thus we get by (2.3) with $y-z$ in the place of y and with $z=0$ that

$$\begin{aligned}
&\int_{B_{S_1}^c \cap B_{S/2}(y)} \int_T^\infty (|y-\tau t e_1 - x|^2+t)^{-3/2-|\alpha|/2+\epsilon}\,dt\,|x|^{-A}\,s_\tau(x)^{-B}\,dx \\
&\le \int_{B_{S_1}^c \cap B_{S/2}(y)} \int_T^\infty \big(\,(|y-\tau t e_1 - x|^2+t)^{-2} + \min\{1,\, t^{-3/2-|\alpha|/2+\epsilon}\}\,\big)\,dt \\
&\qquad\qquad\qquad\qquad |x|^{-A}\,s_\tau(x)^{-B}\,dx \\
&\le \mathfrak{C}(S) \int_{B_{S_1}^c \cap B_{S/2}(y)} \Big(\,|y-x|^{-2} + \int_0^\infty \min\{1,\, t^{-3/2-|\alpha|/2+\epsilon}\}\,\big)\,dt\Big)\,|x|^{-A}\,s_\tau(x)^{-B}\,dx \\
&\le \mathfrak{C}(S,\epsilon) \int_{B_{S_1}^c \cap B_{S/2}(y)} (|y-x|^{-2}+1)\,|x|^{-A}\,s_\tau(x)^{-B}\,dx,
\end{aligned}$$

where we used the assumption $\epsilon < 1/2 + |\alpha|/2$ in the last inequality. On the other hand, we apply (2.6) with $y,\ \nu$ replaced by $y-x,\ -3/2-|\alpha|/2+\epsilon$, respectively, and with $z=0,\ \epsilon=0$, to obtain

$$\int_T^\infty (|y-\tau t e_1 - x|^2+t)^{-3/2-|\alpha|/2+\epsilon}\,dt \le \mathfrak{C}(S,\epsilon)\big(\,|y-x|\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon}$$

for $x \in B_{S_1}^c \backslash B_{S/2}(y)$. Here the assumption $\epsilon < 1/2 + |\alpha|/2$ is again relevant. Now we may deduce from (3.7),

$$\begin{aligned}
\mathfrak{A}_\alpha \le \mathfrak{C}(S,\epsilon)\gamma T^{-\epsilon}\Big(&\int_{B_{S_1}^c \cap B_{S/2}(y)} (|y-x|^{-2}+1)\,|x|^{-A}\,s_\tau(x)^{-B}\,dx \\
&+ \int_{B_{S_1}^c \backslash B_{S/2}(y)} \big(\,|y-x|\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon}\,|x|^{-A}\,s_\tau(x)^{-B}\,dx\Big).
\end{aligned} \tag{3.8}$$

Next we observe that for $x \in B_{S/2}(y)$, we have $|x| \ge |y| - |y-x| \ge |y| - S/2 \ge |y|/2$, and by Lemma 3,

$$s_\tau(x)^{-1} \le \mathfrak{C}(1+|y-x|)\, s_\tau(y)^{-1} \le \mathfrak{C}(S)\, s_\tau(y)^{-1}.$$

For $x \in B_{S/2}(y)^c$, we find

$$\begin{aligned}|y-x| &= |y-x|/2 + |y-x|/2 \ge S/4 + |y-x|/2 \\ &\ge \min\{S/4,\, 1/2\}\,(1+|y-x|),\end{aligned}$$

and for $x \in B^c_{S_1}$ we get $|x| \ge \mathfrak{C}(S_1)\,(1+|x|)$. Therefore from (3.8),

$$\begin{aligned}\mathfrak{A}_\alpha &\le \mathfrak{C}(S,S_1,A,B,\epsilon)\,T^{-\epsilon}\gamma\Big(|y|^{-A}\, s_\tau(y)^{-B}\int_{B_{S/2}(y)}(|y-x|^{-2}+1)\,dx \\ &\quad + \int_{B^c_{S_1}\setminus B_{S/2}(y)}\big(\,(1+|y-x|)\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon}(1+|x|)^{-A}\, s_\tau(x)^{-B}\;dx\Big) \\ &\le \mathfrak{C}(S,S_1,A,B,\epsilon)\,\gamma T^{-\epsilon}\Big(|y|^{-A}\, s_\tau(y)^{-B} \\ &\quad + \int_{\mathbb{R}^3}\big(\,(1+|y-x|)\,s_\tau(y-x)\,\big)^{-1-|\alpha|/2+\epsilon}(1+|x|)^{-A}\, s_\tau(x)^{-B}\;dx\Big).\end{aligned} \tag{3.9}$$

By Lemma 5 and because $y \in B^c_S,\ A - 3/2 > 0,\ A + B \ge A + \min\{1, B\} \ge 3$, we further observe that

$$|y|^{-A}\, s_\tau(y)^{-B} \le \mathfrak{C}(S,A)\,|y|^{-3/2}\, s_\tau(y)^{-A+3/2-B} \le \mathfrak{C}(S,A)\,|y|^{-3/2}\, s_\tau(y)^{-3/2}. \tag{3.10}$$

Moreover, by the proof of [19, Theorem 3.1] we get

$$\int_{\mathbb{R}^3}\big(\,(1+|y-x|)\,s_\tau(y-x)\,\big)^{-1+\epsilon}(1+|x|)^{-A}\, s_\tau(x)^{-B}\;dx \le \mathfrak{C}(\epsilon,A,B)\big(\,|y|\,s_\tau(y)\,\big)^{-1+\epsilon}\, l_{A,B}(y).$$

Similarly, the proof of [19, Theorem 3.2] yields

$$\begin{aligned}&\int_{\mathbb{R}^3}\big(\,(1+|y-x|)\,s_\tau(y-x)\,\big)^{-3/2+\epsilon}(1+|x|)^{-A}\, s_\tau(x)^{-B}\;dx \\ &\le \mathfrak{C}(\epsilon,A,B)\,\big(\,|y|\,s_\tau(y)\,\big)^{-3/2+\epsilon}\, l_{A,B}(y)\, s_\tau(y)^{\max(0,\,7/2-A-B-2\epsilon)}.\end{aligned}$$

The two preceding estimates together with (3.6), (3.9) and (3.10) imply (3.4) and (3.5). □

Corollary 1. *Consider the situation of Theorem 2. As additional assumptions, suppose that $A + \min\{1,B\} > 3$, $T > 0$ and $\epsilon < 1/2$. Then $f \in L^1(\mathbb{R}^3)^3$.*

For $v \in W^{1,1}_{loc}(\mathbb{R}^3)^3$, define

$$\begin{aligned}\|v\|_{1,\infty,w,\epsilon} &:= \sup\{|v(x)|\,[(1+|x|)\, s_\tau(x)]^{1-\epsilon}\;:\; x \in \mathbb{R}^3\} \\ &\quad + \sup\{|\nabla v(x)|\,[(1+|x|)\, s_\tau(x)]^{3/2-\epsilon}\, s_\tau(x)^{-\max(0,\,7/2-A-B-2\epsilon)}\;:\; x \in \mathbb{R}^3\}.\end{aligned}$$

Then

$$\|\mathfrak{R}(f)(\,\cdot\,,T)\|_{1,\infty,w,\epsilon} \le \mathfrak{C}(S,S_1,A,B,\epsilon)\,(\|f\|_1 + \gamma)\,\max\{T^{-\epsilon}, T^{-1}\}.$$

Proof. Put $B^* := \min\{1,B\}$, $\delta := \min\{(A-2)/2,\, (A+B^*-3)/2\}$. The assumption $A+B^* > 3$ implies $A > 2$, so $\delta > 0$ and $-A+2-\delta < 0$. Thus we get with Lemma 5 that

$$|x|^{-A} s_\tau(x)^{-B} = |x|^{-2-\delta}|x|^{-A+2+\delta} s_\tau(x)^{-B} \le \mathfrak{C}(S_1,A)\,|x|^{-2-\delta} s_\tau(x)^{-A-B+2+\delta} \tag{3.11}$$

for $x \in B^c_{S_1}$. We further observe that $-A-B+2+\delta \le -A-B^*+2+\delta < -1$, where the last inequality follows from the choice of δ and the assumption $A+B^* > 3$. Now Lemma 4 and (3.11) yield $\int_{B^c_{S_1}} |f|\,dx < \infty$, so we may conclude $f \in L^1(\mathbb{R}^3)^3$ in view of the assumption $f|B_{S_1} \in L^p(B_{S_1})^3$. At this point (3.3) implies

$$|\partial_y^\alpha \mathfrak{R}(f)(y,T)| \le \mathfrak{C}(S_1)\, T^{-1/2-|\alpha|/2} \|f\|_1 \quad \text{for } y \in B_{S_1},\ \alpha \in \mathbb{N}_0 \text{ with } |\alpha| \le 1. \tag{3.12}$$

Obviously $1 \ge \mathfrak{C}(S_1)\,(1+|y|)\, s_\tau(y)$ for $y \in B_{S_1}$ and $|y| \ge \mathfrak{C}(S_1)\,(1+|y|)$ for $y \in B^c_{S_1}$. Thus Corollary 1 follows from Theorem 2 (in the case $|y| \ge 1$) and inequality (3.12) (in the case $|y| \le 1$). □

4. Comments

Let $f \in C_0^\infty(\mathbb{R}^3)^3$. It is implicit in the proof of [3, Theorem 4.2] that the function $U := \mathfrak{R}(f)(\,\cdot\,,0)$ is the velocity part of a classical solution to the stationary problem (1.4) in the whole space $\mathbb{R}^3$. On the other hand, according to [22, Theorem 1.2], the velocity part u of a solution to (1.3) in $\mathbb{R}^3 \times (0,\infty)$ with initial data zero is given by $u(x,t) := \int_0^t \int_{\mathbb{R}^3} \Gamma(x,z,t-s)\, f(z)\, dz\, ds$.

But $U - u(\,\cdot\,,T) = \mathfrak{R}(f)(\,\cdot\,,T)$ for $T > 0$, so Theorem 2 yields a decay estimate of $U(x) - u(x,t)$ with respect to the space variable x and the time variable t. In addition, the function $\mathfrak{R}(f)(\,\cdot\,,0)$ is known to satisfy all the statements of Theorem 2 with $\epsilon = 0$ ([4, Theorem 3.1]). Therefore these statements with $\epsilon = 0$ carry over to $u(\,\cdot\,,t)$, yielding pointwise spatial decay estimates of $u(\,\cdot\,,t)$ which are uniform with respect to $t \in (0,\infty)$. These estimates are optimal in the sense that the fundamental solution of the stationary Oseen system (without rotational terms) decays with those same rates ([19]). The powers of s_τ appearing in the estimates stated in Theorem 2 should be considered as a mathematical manifestation of the wake extending behind a body which moves in a viscous incompressible fluid.

Corollary 1 means that $U - u(\,\cdot\,,t)$ converges to zero for $t \to \infty$ with respect to the weighted $W^{1,\infty}$-norm $\|\ \|_{1,\infty,w,\epsilon}$. As already mentioned in Section 1, this convergence result means in particular that U is unconditionally asymptotically stable with respect to this norm. The notion of stability which we refer to here is the one introduced in [15, Definition 5.2] in a Hilbert space setting. Obviously it may also be used in the context of Banach spaces.

References

[1] Z. Chen, T. Miyakawa, *Decay properties of weak solutions to a perturbed Navier-Stokes system in R^n*, Adv. Math. Sci. Appl. **7** (1997), 741–770.

[2] P. Deuring, S. Kračmar, *Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: approximation by flows in bounded domains,* Math. Nachr. **269–270** (2004), 86–115.

[3] P. Deuring, S. Kračmar, Š. Nečasová, *A representation formula for linearized stationary incompressible viscous flows around rotating and translating bodies,* Discrete Contin. Dyn. Syst. Ser. S **3** (2010), 237–253.

[4] P. Deuring, S. Kračmar, Š. Nečasová, *On pointwise decay of linearized stationary incompressible viscous flow around rotating and translating bodies*, SIAM J. Math. Anal. **43** (2011), 705–738.

[5] P. Deuring, S. Kračmar, Š. Nečasová, *Linearized stationary incompressible flow around rotating and translating bodies: asymptotic profile of the velocity gradient and decay estimate of the second derivatives of the velocity,* J. Differential Equations **252** (2012), 459–476.

[6] P. Deuring, S. Kračmar, Š. Nečasová, *A linearized system describing stationary incompressible viscous flow around rotating and translating bodies: improved decay estimates of the velocity and its gradient,* In: Dynamical Systems, Differential Equations and Applications, Vol. I, Ed. by W. Feng, Z. Feng, M. Grasselli, A. Ibragimov, X. Lu, S. Siegmund and J. Voigt. Discrete Contin. Dyn. Syst. (Supplement 2011, 8th AIMS Conference, Dresden, Germany) (2011), 351–361.

[7] P. Deuring, S. Kračmar, Š. Nečasová, *Pointwise decay of stationary rotational viscous incompressible flows with nonzero velocity at infinity,* J. Differential Equations **255** (2013), 1576–1606.

[8] P. Deuring, S. Kračmar, Š. Nečasová, *Linearized stationary incompressible flow around rotating and translating bodies - Leray solutions,* Discrete Contin. Dyn. Syst. Ser. S **7** (2014), 967–979.

[9] R. Farwig, *The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces,* Math. Z. **211** (1992), 409–447.

[10] R. Farwig, *An L^q-analysis of viscous fluid flow past a rotating obstacle,* Tôhoku Math. J. **58** (2006), 129–147.

[11] R. Farwig, T. Hishida, D. Müller, *L^q-Theory of a singular "winding" integral operator arising from fluid dynamics,* Pacific J. Math. **215** (2004), 297–312.

[12] R. Farwig, M. Krbec, Š. Nečasová, *A weighted L^q - approach to Oseen flow around a rotating body,* Math. Methods Appl. Sci. **31** (2008), 551–574.

[13] G.P. Galdi, *On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications,* Handbook of Mathematical Fluid Dynamics, Vol. 1, Ed. by S. Friedlander, D. Serre, Elsevier, 2002.

[14] G.P. Galdi, M. Kyed, *A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part I: Strong solutions,* Proc. Amer. Math. Soc. **141** (2013), 573–583.

[15] G.P. Galdi, S. Rionero, *Weighted Energy Methods in Fluid Dynamics and Elasticity,* Lecture Notes in Mathematics **1134**, Springer, Berlin e.a., 1985.

[16] G.P. Galdi, A.L. Silvestre, *The steady motion of a Navier-Stokes liquid around a rigid body,* Arch. Rat. Mech. Anal. **184** (2007), 371–400.

[17] T. Hishida, *L^q estimates of weak solutions to the stationary Stokes equations around a rotating body,* J. Math. Soc. Japan **58** (2006), 743–767.

[18] S. Kračmar, Š. Nečasová, P. Penel, *L^q - approach of weak solutions of Oseen flow around a rotating body,* Quart. of Appl. Math. **68** (2010), 421–437.

[19] S. Kračmar, A. Novotný, M. Pokorný, *Estimates of Oseen kernels in weighted L^p spaces,* J. Math. Soc. Japan **53** (2001), 59–111.

[20] M. Kyed, *On a mapping property of the Oseen operator with rotation*, Discrete Contin. Dyn. Syst. Ser. S **6** (2013), 1315–1322.

[21] Š. Nečasová, K. Schumacher, *Strong solution of the Stokes flow around a rotating body in weighted L^q-spaces,* Math. Nachr. **284** (2011), 1701–1714.

[22] E.A. Thomann, R.B. Guenther, *The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions – time dependent case,* J. Math. Fluid Mech. **8** (2006), 77–98.

Univ. Littoral Côte d'Opale, Laboratoire de mathématiques pures et appliquées Joseph Liouville, F-62228 Calais, France

E-mail address: Paul.Deuring@lmpa.univ-littoral.fr

Department of Technical Mathematics, Czech Technical University, Karlovo nám. 13, 121 35 Praha 2, Czech Republic, and Mathematical Institute of the Academy of Sciences of the Czech Republic Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: Stanislav.Kracmar@fs.cvut.cz

Mathematical Institute of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: matus@math.cas.cz

University of Geneva 24, Quai Ernest Ansermet 1205 Geneva - Switzerland

E-mail address: Peter.Wittwer@unige.ch

JEPE Vol 1, 2015, p. 335-362

Orthogonal $\perp$

POSITIVE SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH SINGULAR DIRICHLET BOUNDARY DATA

MAREK FILA, KAZUHIRO ISHIGE, AND TATSUKI KAWAKAMI

Abstract. The purpose of this paper is to construct positive solutions of the semilinear elliptic equation $-\Delta u = u^p$ in $\mathbb{R}^N_+$ with a singular Dirichlet boundary condition. We show that for $p > (N+1)/(N-1)$ there exists a positive singular solution which behaves like $|x|^{-2/(p-1)}$ as $|x| \to 0$ and like the Poisson kernel as $|x| \to \infty$.

1. Introduction

We consider the problem

$$
\begin{cases}
-\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \\
u > 0 & \text{in} \quad \mathbb{R}^N_+, \\
u = \varphi \geq 0 & \text{on} \quad \partial\mathbb{R}^N_+,
\end{cases} \tag{1.1}
$$

where $N \geq 2$, $\mathbb{R}^N_+ := \{x = (x', x_N) \, : \, x' \in \mathbb{R}^{N-1}, \ x_N > 0\}$, $p > 1$ and φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$.

For problem (1.1), the exponent

$$p_* := (N+1)/(N-1)$$

plays an important role. Namely, it was shown in [2] that for $p \leq p_*$ there is no classical solution of (1.1), no matter which boundary data φ we impose. As we shall see below, this result is sharp.

If $\varphi \equiv 0$ then there are Liouville-type theorems saying that no classical solution of (1.1) exists if $(N-2)p < N+2$ (see [12]) and no classical bounded solution of (1.1) exists if $p < p_c$

2010 *Mathematics Subject Classification.* Primary: 35J61; Secondary: 35B40, 35J25.

Key words and phrases. Semilinear elliptic equation, singular Dirichlet boundary data, asymptotic behavior of solutions.

Received 18/01/2016, accepted 20/01/2016.

The first author was supported by the Slovak Research and Development Agency under the contract No. APVV-14-0378 and by the VEGA grant 1/0711/12. The second author was supported by the Grant-in-Aid for Scientific Research (A)(No. 15H02058), from Japan Society for the Promotion of Science. The third author was supported by the Grant-in-Aid for Young Scientists (B) (No. 24740107) from Japan Society for the Promotion of Science and by the JSPS Program for Advancing Strategic International Networks to Accelerate the Circulation of Talented Researchers "Mathematical Science of Symmetry, Topology and Moduli, Evolution of International Research Network based on OCAMI".

(see [7]), here

$$p_c := \begin{cases} \infty, & N \le 11, \\ \dfrac{(N-3)^2 - 4N + 4 + 8\sqrt{N-2}}{(N-3)(N-11)}, & N > 11. \end{cases}$$

If $p > p_*$ then a solution of (1.1) exists when $\varphi \not\equiv 0$ is small enough. More precisely, if ψ is bounded, $\psi \not\equiv 0$ and

$$\limsup_{|x'| \to \infty} |x'|^{2/(p-1)} \psi(x') < \infty$$

then there exists a solution (in the sense of Definition 1.1 below) when $\varphi = k\psi$ and $k > 0$ is small enough (see Corollary 1.3 in [11]). Also, when $p > p_*$, a different condition for $\varphi \in L^1(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1})$, $\varphi \not\equiv 0$, which guarantees the existence of a solution of (1.1), follows from Theorem 1.4 in [11] and Theorem 1.2 in [8]. Namely, it is sufficient if

$$\|\varphi\|_{L^1(\mathbb{R}^{N-1})} \|\varphi\|_{L^\infty(\mathbb{R}^{N-1})}^{(Np-p-N-1)/2}$$

is small enough.

Singular solutions of the problem

$$\begin{cases} -\Delta u = u^p & \text{in} \quad \mathbb{R}^N_+, \qquad p > 1, \\ u > 0 & \text{in} \quad \mathbb{R}^N_+, \\ u = 0 & \text{on} \quad \partial\mathbb{R}^N_+ \setminus \{0\}, \end{cases} \tag{1.2}$$

were studied in [3] and [6]. It was shown in [3] that a solution of the form

$$u_*(x) := |x|^{-\frac{2}{p-1}} \omega\left(\frac{x}{|x|}\right)$$

exists if and only if

$$p_* < p < p^* := \begin{cases} \infty, & N \le 3, \\ \dfrac{N+1}{N-3}, & N > 3. \end{cases}$$

The existence of a different singular solution U of (1.2) was established in [6] for $p \in (p_*, p_* + \varepsilon)$ where $\varepsilon > 0$. This solution U behaves like u_* as $x \to 0$ and like the Poisson kernel as $|x| \to \infty$.

In this paper we prove that for $p > p_*$ and suitable singular boundary data φ there are singular solutions of (1.1) which behave like $|x|^{-2/(p-1)}$ as $|x| \to 0$ and like the Poisson kernel as $|x| \to \infty$.

For $p > N/(N-2)$, $N > 2$, there is an explicit singular solution

$$u_\infty := c_{p,N} |x|^{-\frac{2}{p-1}}, \qquad c_{p,N} := \left(\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right)^{\frac{1}{p-1}},$$

and for $N/(N-2) < p < (N+2)/(N-2)$, $N > 2$, there is a family of radial singular solutions u_α, $\alpha > 0$, such that

$$\lim_{|x| \to \infty} |x|^{N-2} u_\alpha(x) = \alpha,$$

see [5]. The solutions we find behave differently from u_α, $0 < \alpha \le \infty$, as $|x| \to \infty$.

For other works on boundary singularities of solutions of semilinear elliptic equations we refer to [4, 13, 15, 16, 17, 18].

To formulate our results we introduce some notation. For any $x' \in \mathbb{R}^{N-1}$ and $\lambda > 0$, let P be the $(N-1)$-dimensional Poisson kernel, that is,

$$P(x', \lambda) := c_N \lambda \left(\lambda^2 + |x'|^2\right)^{-\frac{N}{2}}, \tag{1.3}$$

for $x' \in \mathbb{R}^{N-1}$ and $\lambda > 0$, where c_N is a constant chosen so that

$$\int_{\mathbb{R}^{N-1}} P(x', \lambda)\, dx' = 1, \qquad \lambda > 0. \tag{1.4}$$

Throughout this paper, we often identify $\mathbb{R}^{N-1}$ with $\partial\mathbb{R}^N_+$. Next we set $y_* := (y', -y_N)$ for $y = (y', y_N) \in \mathbb{R}^N_+$ and

$$G(x,y) := \begin{cases} \dfrac{c_N}{2(N-2)} \left(|x-y|^{-(N-2)} - |x-y_*|^{-(N-2)}\right) & \text{if} \quad N \geq 3, \\ \dfrac{1}{4\pi} \log\left(1 + \dfrac{4x_2 y_2}{|x-y|^2}\right) & \text{if} \quad N = 2, \end{cases} \tag{1.5}$$

which is the Green function for $-\Delta_D$. Here Δ_D is the Laplace operator in $\mathbb{R}^N_+$ with the homogeneous Dirichlet boundary condition.

Definition 1.1. Let φ be a nonnegative measurable function in $\mathbb{R}^{N-1}$. Let u be a nonnegative measurable function in $\mathbb{R}^N_+$.
(i) We call the function u a solution of (1.1) if u satisfies

$$u(x', x_N) = \int_{\mathbb{R}^{N-1}} P(x'-y', x_N)\varphi(y')\, dy' + \int_{\mathbb{R}^N_+} G(x,y) u(y)^p\, dy < \infty$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \in [0, \infty)$.
(ii) We call u a minimal solution of (1.1) if, for any solution v of (1.1),

$$u(x', x_N) \leq v(x', x_N)$$

holds for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$.

Now we are ready to state the main result of this paper.

Theorem 1.1. *Let $N \geq 2$ and $p > p_* = (N+1)/(N-1)$. Put*

$$\psi(x') = \min\{|x'|^{-\frac{2}{p-1}}, |x'|^{-N}\} = \begin{cases} |x'|^{-\frac{2}{p-1}} & \text{if} \quad |x'| \leq 1, \\ |x'|^{-N} & \text{if} \quad |x'| > 1, \end{cases} \tag{1.6}$$

for $x' \in \mathbb{R}^{N-1}$. Then there exist constants $0 < c < k < 1$ such that if

$$c\psi(x') \leq \varphi(x') \leq k\psi(x'), \qquad x' \in \mathbb{R}^{N-1}, \tag{1.7}$$

then problem (1.1) possesses a minimal solution u. Furthermore,

$$K|x|^{-\frac{2}{p-1}} \leq u(x) \leq L|x|^{-\frac{2}{p-1}}, \qquad x \in D_{\text{in}} := B(0,1) \cap \mathbb{R}^N_+, \tag{1.8}$$

and

$$K(1+x_N)|x|^{-N} \leq u(x) \leq L(1+x_N)|x|^{-N}, \qquad x \in D_{\text{out}} := \mathbb{R}^N_+ \setminus \overline{B(0,1)}, \tag{1.9}$$

for some positive constants K and L with $K < L$.

The rest of this paper is organized as follows. In Section 2 we recall some preliminary inequalities. In Section 3 we give pointwise estimates of $S(x_N)\psi$. In Section 4 we obtain upper estimates of an integral associated with the Green function G and prove Theorem 1.1.

2. Preliminaries

We recall some properties of the semigroup associated with the Poisson kernel P and a proposition on the Green function G. Here and in the rest of the paper, by c and C we denote generic positive constants which may have different values also within the same line.

We introduce some notation. For any $R > 0$, let

$$B(0,R) := \{x \in \mathbb{R}^N : |x| < R\} \quad \text{and} \quad B'(0,R) := \{x' \in \mathbb{R}^{N-1} : |x'| < R\}.$$

For any $1 \le r \le \infty$, we denote by $\|\cdot\|_r$ the usual norm of $L^r := L^r(\mathbb{R}^{N-1})$. For any measurable set E in $\mathbb{R}^{N-1}$, we denote by $|E|$ the $N-1$ dimensional Lebesgue measure of E. For any measurable function f in $\mathbb{R}^{N-1}$, let

$$\mu_f(\lambda) := |\{x : |f(x)| > \lambda\}| \qquad (\lambda \ge 0)$$

be the distribution function of f. We define the non-increasing rearrangement of f by

$$f^*(s) := \inf\{\lambda > 0 : \mu_f(\lambda) \le s\}.$$

For any $1 \le r \le \infty$, we define the $L^{r,\infty}$ space by

$$L^{r,\infty} := \{f : f \text{ is measurable in } \mathbb{R}^{N-1},\ \|f\|_{r,\infty} < \infty\},$$

where

$$\|f\|_{r,\infty} := \sup_{s>0} s^{\frac{1}{r}} f^{**}(s), \qquad f^{**}(s) := \frac{1}{s}\int_0^s f^*(r)dr.$$

For any measurable function φ in $\mathbb{R}^{N-1}$, let

$$[S(x_N)\varphi](x') := \int_{\mathbb{R}^{N-1}} P(x'-y', x_N)\varphi(y')\,dy' \tag{2.1}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$. Then, for any $\lambda > 0$,

$$[S(\lambda + x_N)\varphi](x') = [S(\lambda)(S(x_N)\varphi)](x'),$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \ge 0$ if either φ is a nonnegative measurable function in $\mathbb{R}^{N-1}$ or $\varphi \in L^{q,\infty}$ for some $q \in [1,\infty]$. In particular, for any $\varphi \in L^{q,\infty}$ and $1 \le q \le \infty$, there exists a constant C such that

$$\|S(2\lambda)\varphi\|_\infty \le \|P(\lambda)\|_p \|S(\lambda)\varphi\|_r \le C\|P(\lambda)\|_p\|P(\lambda)\|_s\|\varphi\|_{q,\infty}, \qquad \lambda > 0, \tag{2.2}$$

where $1 < p, r, s < \infty$ satisfy

$$\frac{1}{p} + \frac{1}{r} = 1, \qquad \frac{1}{s} + \frac{1}{q} = 1 + \frac{1}{r}$$

(see [14, Section 2]). By (1.3) and (1.4), for any $1 \le r \le \infty$, we have

$$\|P(\lambda)\|_r \le C\lambda^{-(N-1)(1-\frac{1}{r})}, \qquad \lambda > 0.$$

This together with (2.2) yields

$$\|S(\lambda)\varphi\|_\infty \le C\lambda^{-\frac{N-1}{q}}\|\varphi\|_{q,\infty}, \qquad \lambda > 0, \qquad q \in [1,\infty]. \tag{2.3}$$

Next we recall some estimates of the Green function G (see [1, Proposition 1] and [9]).

Proposition 2.1. *Let G be as in* (1.5)*.*

(i) *If $N \geq 3$ then there exists a positive constant C such that*

$$G(x,y) \leq C \frac{x_N y_N}{|x-y|^{N-2}(|x-y|^2 + 4x_N y_N)}, \qquad x, y \in \mathbb{R}^N_+, \quad x \neq y. \tag{2.4}$$

(ii) *If $N = 2$ then, for any $\alpha \in (0,1]$, there exists a positive constant C such that*

$$G(x,y) \leq C \left(\frac{x_2 y_2}{|x-y|^2} \right)^{\alpha}, \qquad x, y \in \mathbb{R}^2_+, \quad x \neq y. \tag{2.5}$$

3. Pointwise estimates of harmonic extensions

In this section we obtain some pointwise estimates of $S(x_N)\psi$. In what follows, we set

$$m := \frac{2}{p-1}$$

for simplicity. Then

$$m < N - 1 \quad \text{if} \quad p > p_*. \tag{3.1}$$

Furthermore, let D_{in} and D_{out} are as in (1.8) and (1.9), respectively.

Lemma 3.1. *Let $p > p_*$ and ψ be as in* (1.6)*. Then there exists a positive constant C such that*

$$[S(x_N)\psi](x') \geq C|x|^{-m}, \quad x \in D_{\mathrm{in}}, \tag{3.2}$$

$$[S(x_N)\psi](x') \geq C(1 + x_N)|x|^{-N}, \quad x \in D_{\mathrm{out}}. \tag{3.3}$$

Proof. It follows from (1.3) and (2.1) that

$$\begin{aligned} [S(x_N)\psi](x') &= c_N x_N^{-(N-1)} \int_{\mathbb{R}^{N-1}} \left(1 + \left| \frac{y'}{x_N} \right|^2 \right)^{-\frac{N}{2}} \psi(x' - y')\, dy' \\ &= c_N \int_{\mathbb{R}^{N-1}} \left(1 + |z'|^2 \right)^{-\frac{N}{2}} \psi(x' - x_N z')\, dz', \quad x \in \mathbb{R}^N_+. \end{aligned} \tag{3.4}$$

We prove (3.2). For any $x \in D_{\mathrm{in}}$, since

$$|x' - x_N z'| \leq |x'| + x_N |z'| \leq |x'| + x_N \leq \sqrt{2}, \qquad z' \in B'(0,1),$$

by (1.6) and (3.4) we have

$$\begin{aligned} [S(x_N)\psi](x') &\geq C \int_{\mathbb{R}^{N-1}} (1 + |z'|)^{-N} \psi(x' - x_N z') dy' \\ &\geq C \int_{B'(0,1)} (1 + |z'|)^{-N} |x' - x_N z'|^{-m}\, dy' \\ &\geq C(|x'| + x_N)^{-m} \int_0^1 (1+r)^{-N} r^{N-2} dr \geq C|x|^{-m}. \end{aligned}$$

This implies (3.2).

We prove (3.3). Set

$$D_1 := \left\{ x \in D_{\rm out} : x_N \le \frac{1}{4} \right\}, \quad D_2 := \left\{ x \in D_{\rm out} : x_N > \frac{1}{4} \right\}.$$

For any $x \in D_1$, since

$$\frac{\sqrt{3}-1}{2} \le |x'| - 2x_N \le |x'| - x_N|z'| \le |x' - x_N z'| \le |x'| + x_N|z'| \le 2(|x'| + x_N)$$

for all $z' \in \mathbb{R}^{N-1}$ with $1 \le |z'| \le 2$, it follows from (1.6) and (3.4) that

$$\begin{aligned} [S(x_N)\psi](x') &\ge C \int_{\mathbb{R}^{N-1}} (1+|z'|)^{-N} \psi(x' - x_N z') dy' \\ &\ge C \int_{1 \le |z'| \le 2} (1+|z'|)^{-N} |x' - x_N z'|^{-N} dy' \\ &\ge C|x|^{-N} \ge C(1+x_N)|x|^{-N}, \qquad x \in D_1. \end{aligned} \tag{3.5}$$

On the other hand, for any $x \in D_2$, since $5x_N > 1 + x_N$, by (1.3), (1.6) and (2.1) we have

$$\begin{aligned} [S(x_N)\psi](x') &\ge C x_N \int_{1 \le |y'| \le 2} (x_N + |x'| + |y'|)^{-N} |y'|^{-N} dy' \\ &\ge C(1+x_N)|x|^{-N} \int_1^2 r^{-2} dr \ge C(1+x_N)|x|^{-N}, \qquad x \in D_2. \end{aligned}$$

This together with (3.5) implies (3.3), and the proof is complete. ☐

Lemma 3.2. *Let $p > p_*$ and ψ be as in* (1.6). *Then there exists a positive constant C such that*

$$[S(x_N)\psi](x') \le C|x|^{-m}, \quad x \in D_{\rm in}, \tag{3.6}$$

$$[S(x_N)\psi](x') \le C(1+x_N)|x|^{-N}, \quad x \in D_{\rm out}. \tag{3.7}$$

Proof. We first prove (3.6). Let $x \in D_{\rm in}$ and set

$$B_1(x') := \left\{ y' \in \mathbb{R}^{N-1} : |x' - y'| \le \frac{|x'|}{2} \right\}, \quad B_2(x') := \left\{ y' \in \mathbb{R}^{N-1} : |x' - y'| > \frac{|x'|}{2} \right\}.$$

It follows from (1.3) and (2.1) that

$$\begin{aligned} [S(x_N)\psi](x') &= c_N x_N \int_{\mathbb{R}^{N-1}} \left(x_N^2 + |y'|^2 \right)^{-\frac{N}{2}} \psi(x' - y')\, dy' \\ &\le C \left(\int_{B_1(x')} + \int_{B_2(x')} \right) x_N (x_N + |y'|)^{-N} \psi(x' - y')\, dy'. \end{aligned} \tag{3.8}$$

By (3.1) we can find $\varepsilon \in (0,1)$ such that

$$-(N-1) + m + \varepsilon < 0. \tag{3.9}$$

For any $y' \in B_1(x')$, since $|x'|/2 \le |y'| \le (3|x'|)/2$, we see that

$$x_N (x_N + |y'|)^{-N} \le C|x'|^{-(N-1)} \le C|x'|^{-m-\varepsilon} |x' - y'|^{-(N-1)+m+\varepsilon}. \tag{3.10}$$

By (1.6) and (3.10) we have

$$\begin{aligned}\int_{B_1(x')} x_N(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le \int_{B_1(x')} x_N(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\ &\le C|x'|^{-m-\varepsilon}\int_{B_1(x')}|x'-y'|^{-(N-1)+\varepsilon}\,dy'\\ &= C|x'|^{-m-\varepsilon}\int_0^{\frac{|x'|}{2}} r^{-1+\varepsilon}\,dr \le C|x'|^{-m}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{B_2(x')} x_N(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &= \int_{B_2(x')} x_N(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\ &\le C|x'|^{-m}\int_{\mathbb{R}^{N-1}}(1+|z'|)^{-N}\,dz' \le C|x'|^{-m}.\end{aligned}$$

These together with (3.8) yield

$$[S(x_N)\psi](x') \le C|x'|^{-m}, \qquad x\in D_{\mathrm{in}}. \tag{3.11}$$

On the other hand, it follows from (1.6) and (3.1) that $\psi\in L^{r_*,\infty}$ with $r_*=(N-1)/m>1$. This together with (2.3) implies that

$$[S(x_N)\psi](x') \le \|S(x_N)\psi\|_\infty \le Cx_N^{-m}\|\psi\|_{r_*,\infty}, \quad x\in\mathbb{R}^N_+. \tag{3.12}$$

Therefore, by (3.11) and (3.12) we obtain

$$[S(x_N)\psi](x') \le C\min\left\{|x'|^{-m}, x_N^{-m}\right\}, \qquad x\in D_{\mathrm{in}}. \tag{3.13}$$

On the other hand, the following inequality holds (see e.g., [10, Section 4]): For any $q>0$, there exists a positive constant C such that

$$\min\left\{a^{-q}, b^{-q}\right\} \le C(a+b)^{-q} \quad \text{for} \quad a,b>0. \tag{3.14}$$

Then, by (3.13) and (3.14) we have (3.6).

Next we prove (3.7). Let $x\in D_{\mathrm{out}}$ with $|x'|\le 1/2$ and set

$$B_3(x') := \left\{y'\in\mathbb{R}^{N-1} : |x'-y'|\le 2\right\}, \quad B_4(x') := \{y'\in\mathbb{R}^{N-1} : |x'-y'|>2\}.$$

Similarly to (3.8), we have

$$[S(x_N)\psi](x') \le Cx_N\left(\int_{B_3(x')}+\int_{B_4(x')}\right)(x_N+|y'|)^{-N}\psi(x'-y')\,dy'. \tag{3.15}$$

Since $2x_N\ge\sqrt{3}>1\ge|x'|$, we see that $|x|\le Cx_N$. It follows from (1.6) and (3.1) that

$$\begin{aligned}x_N\int_{B_3(x')}(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le x_N\int_{B_3(x')}(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\ &\le x_N^{-N+1}\int_{B_3(x')}|x'-y'|^{-m}\,dy'\\ &\le C(1+x_N)|x|^{-N}\int_0^2 r^{-m+N-2}\,dr \le C(1+x_N)|x|^{-N}.\end{aligned} \tag{3.16}$$

Furthermore, since $|x'|\le 1/2$ and

$$|y'| \ge |x'-y'|-|x'| \ge |x'-y'|-1 \ge 1, \qquad y'\in B_4(x'),$$

by (1.6) we obtain

$$
\begin{aligned}
x_N\int_{B_4(x')}(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le x_N\int_{B_4(x')}(x_N+|y'|)^{-N}|x'-y'|^{-N}\,dy'\\
&\le (1+x_N)^{-N+1}\int_{B_4(x')}|x'-y'|^{-N}\,dy'\\
&\le C(1+x_N)|x|^{-N}\int_2^\infty r^{-2}\,dr \le C(1+x_N)|x|^{-N}.
\end{aligned}\tag{3.17}
$$

Then, by (3.15), (3.16) and (3.17) we have (3.7) for the case $|x'|\le 1/2$.

It remains to prove (3.7) for the case $|x'|>1/2$. Let $x\in D_{\rm out}$ with $|x'|>1/2$ and set

$$
B_5(x'):=\left\{y'\in\mathbb{R}^{N-1}:|x'-y'|\le\frac{1}{8}\right\},\quad B_6(x'):=\left\{y'\in\mathbb{R}^{N-1}:\frac{1}{8}<|x'-y'|\le\frac{|x'|}{2}\right\},
$$
$$
B_7(x'):=\left\{y'\in\mathbb{R}^{N-1}:|x'-y'|>\frac{|x'|}{2}\right\}.
$$

Similarly to (3.8), we have

$$
[S(x_N)\psi](x')\le Cx_N\left(\int_{B_5(x')}+\int_{B_6(x')}+\int_{B_7(x')}\right)(x_N+|y'|)^{-N}\psi(x'-y')\,dy'.\tag{3.18}
$$

Since $|y'|\ge|x'|-1/8$ for any $y'\in B_5(x')$, by (1.6) and (3.1) we have

$$
\begin{aligned}
x_N\int_{B_5(x')}(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le x_N\int_{B_5(x')}(x_N+|y'|)^{-N}|x'-y'|^{-m}\,dy'\\
&\le C(1+x_N)\int_0^{\frac{1}{8}}\left(x_N+|x'|-\frac{1}{8}\right)^{-N}r^{-m+N-2}\,dr\le C(1+x_N)|x|^{-N}.
\end{aligned}\tag{3.19}
$$

Since $|x'|/2\le|y'|\le 3|x'|/2$ for any $y'\in B_6(x')$, it follows from (1.6) that

$$
\begin{aligned}
x_N\int_{B_6(x')}(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le x_N\int_{B_6(x')}(x_N+|y'|)^{-N}|x'-y'|^{-N}\,dy'\\
&\le C(1+x_N)|x|^{-N}\int_{B_6(x')}|x'-y'|^{-N}\,dy'\\
&\le C(1+x_N)|x|^{-N}\int_{\frac{1}{8}}^\infty r^{-2}\,dr\le C(1+x_N)|x|^{-N}.
\end{aligned}\tag{3.20}
$$

Furthermore, since $|x'-y'|>|x'|/2>1/4$ for any $y'\in B_7(x')$, by (1.6) we obtain

$$
\begin{aligned}
x_N\int_{B_7(x')}(x_N+|y'|)^{-N}\psi(x'-y')\,dy' &\le x_N^{1-N}\int_{B_7(x')}\left(1+\frac{|y'|}{x_N}\right)^{-N}|x'-y'|^{-N}\,dy'\\
&\le C|x'|^{-N}\int_{\mathbb{R}^{N-1}}(1+|z'|)^{-N}\,dz'\le C|x'|^{-N}\le C(1+x_N)|x'|^{-N}.
\end{aligned}\tag{3.21}
$$

Similarly, we see that

$$
\begin{aligned}
x_N \int_{B_7(x')} (x_N + |y'|)^{-N} \psi(x' - y')\, dy' &\le x_N^{1-N} \int_{B_7(x')} \left(1 + \frac{|y'|}{x_N}\right)^{-N} |x' - y'|^{-N}\, dy' \\
&\le (1 + x_N) x_N^{-N} \int_{B_7(x')} |x' - y'|^{-N}\, dy' \qquad (3.22)\\
&\le (1 + x_N) x_N^{-N} \int_{\frac{1}{4}}^{\infty} r^{-2}\, dr \le C(1 + x_N) x_N^{-N}.
\end{aligned}
$$

Then (3.14), (3.21) and (3.22) yield

$$
\begin{aligned}
&x_N \int_{B_7(x')} (x_N + |y'|)^{-N} \psi(x' - y')\, dy' \\
&\le C(1 + x_N) \min\{|x'|^{-N}, x_N^{-N}\} \le C(1 + x_N)|x|^{-N}.
\end{aligned} \tag{3.23}
$$

Then we deduce from (3.18), (3.19), (3.20) and (3.23) that (3.7) holds for the case $|x'| > 1/2$. Thus Lemma 3.2 follows. □

4. Proof of Theorem 1.1

In this section we obtain two pointwise estimates of

$$
[(-\Delta_D)^{-1} f^p](x) := \int_{\mathbb{R}^N_+} G(x, y) f(y)^p\, dy, \qquad x \in \mathbb{R}^N_+, \tag{4.1}
$$

and prove Theorem 1.1.

Lemma 4.1. *Let $p > p_*$. Assume that*

$$
0 \le f(x) \le |x|^{-m}, \qquad x \in \mathbb{R}^N_+. \tag{4.2}
$$

Then there exists a positive constant C such that

$$
[(-\Delta_D)^{-1} f^p](x) \le C x_N |x|^{-m-1}, \qquad x \in \mathbb{R}^N_+.
$$

Proof. Let $x = (x', x_N) \in \mathbb{R}^N_+$. We divide $\mathbb{R}^{N-1}$ into the following three sets:

$$
\begin{aligned}
&D_1 := \left\{ y' \in \mathbb{R}^{N-1} : |x' - y'| \le \frac{|x'|}{2} \right\}, \ D_2 := \left\{ y' \in \mathbb{R}^{N-1} : \frac{|x'|}{2} < |x' - y'| \le 2|x'| \right\}, \\
&D_3 := \{ y' \in \mathbb{R}^{N-1} : |x' - y'| > 2|x'| \}.
\end{aligned}
$$

Since $mp = m + 2$, by (4.1) and (4.2) it suffices to prove that

$$
\int_0^{\infty} \int_{D_k} G(x, y) |y|^{-(m+2)}\, dy'\, dy_N \le C x_N |x|^{-m-1}, \quad k = 1, 2, 3. \tag{4.3}
$$

Here C is independent of $x \in \mathbb{R}^N_+$.

Proof in the case $k = 1$. In the case $|x'| = 0$, we immediately obtain (4.3) in the case $k = 1$. So it suffices to consider the case $|x'| \ne 0$.

For any $y' \in D_1$ we see that $|y'| \ge |x'|/2$. In the case $N \ge 3$, by (2.4) we have

$$\begin{aligned}
&\int_{\frac{x_N}{2}}^\infty \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{x_N}{2}}^\infty \int_{|x'-y'|\le\frac{|x'|}{2}} |x'-y'|^{-(N-2)}|y|^{-(m+2)}\,dy'\,dy_N \\
&\le C\int_{\frac{x_N}{2}}^\infty (|x'|+y_N)^{-(m+2)}\,dy_N \int_{|x'-y'|\le\frac{|x'|}{2}} |x'-y'|^{-(N-2)}\,dy' \\
&\le C(|x'|+x_N)^{-(m+1)}|x'| \le C(|x'|+x_N)^{-m} \le Cx_N^{-m}.
\end{aligned} \tag{4.4}$$

In the case $N=2$, by (2.5) with $\alpha = 1/4$ we have

$$\begin{aligned}
&\int_{\frac{x_2}{2}}^\infty \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C\int_{\frac{x_2}{2}}^\infty \int_{|x_1-y_1|\le\frac{|x_1|}{2}} \left(\frac{x_2y_2}{|x_1-y_1|^2}\right)^{\frac14} (|y_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \\
&\le Cx_2^{\frac14}\int_{\frac{x_2}{2}}^\infty (|x_1|+y_2)^{-m-\frac74}\,dy_2 \int_{|x_1-y_1|\le\frac{|x_1|}{2}} |x_1-y_1|^{-\frac12}\,dy_1 \\
&\le Cx_2^{\frac14}(|x_1|+x_2)^{-m-\frac34}|x_1|^{\frac12} \le Cx_2^{\frac14}(|x_1|+x_2)^{-m-\frac14} \le Cx_2^{-m}.
\end{aligned} \tag{4.5}$$

On the other hand, in the case $N \ge 2$, by (2.4), (2.5) and (3.1) we see that

$$G(x,y) \le C\frac{x_Ny_N}{|x_N-y_N|^{1+m}|x'-y'|^{N-1-m}} \le Cx_N^{-m}\frac{y_N}{|x'-y'|^{N-1-m}} \tag{4.6}$$

for all $y \in \mathbb{R}^N_+$ with $0 \le y_N \le x_N/2$. Then we have

$$\begin{aligned}
&\int_0^{\frac{x_N}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_N^{-m}\int_0^{\frac{x_N}{2}} \int_{|x'-y'|\le\frac{|x'|}{2}} \frac{y_N}{|x'-y'|^{N-1-m}}|y|^{-(m+2)}\,dy'\,dy_N \\
&\le Cx_N^{-m}\int_0^{\frac{x_N}{2}} (|x'|+y_N)^{-(m+1)}\,dy_N \int_{|x'-y'|\le\frac{|x'|}{2}} |x'-y'|^{m+1-N}\,dy' \\
&\le Cx_N^{-m}|x'|^{-m}|x'|^m \le Cx_N^{-m}.
\end{aligned}$$

This together with (4.4) and (4.5) implies that

$$\int_0^\infty \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N^{-m}. \tag{4.7}$$

Next we prove that

$$\int_0^\infty \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N|x'|^{-m-1}. \tag{4.8}$$

In the case $N \geq 3$, similarly to (4.4), by (2.4) we obtain

$$
\begin{aligned}
&\int_0^{\frac{3x_N}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\leq C \int_0^{\frac{3x_N}{2}} \int_{|x'-y'|\leq\frac{|x'|}{2}} |x'-y'|^{-(N-2)}(|y'|+y_N)^{-(m+2)}\,dy'\,dy_N \\
&\leq C|x'| \int_0^{\frac{3x_N}{2}} (|x'|+y_N)^{-(m+2)}\,dy_N \leq Cx_N|x'|^{-m-1}.
\end{aligned} \tag{4.9}
$$

In the case $N = 2$, it follows from (2.5) with $\alpha = 1$ that

$$
G(x,y) \leq C\frac{x_2 y_2}{|x_2-y_2|^2} \leq C \tag{4.10}
$$

for all $y \in \mathbb{R}^2_+$ with $0 \leq y_2 \leq x_2/2$. This together with (4.2) yields

$$
\begin{aligned}
\int_0^{\frac{x_2}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy &\leq C \int_0^{\frac{x_2}{2}} \int_{|x_1-y_1|\leq\frac{|x_1|}{2}} (|y_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \\
&\leq C|x_1| \int_0^{\frac{x_2}{2}} (|x_1|+y_2)^{-(m+2)}\,dy_2 \leq Cx_2|x_1|^{-m-1}.
\end{aligned} \tag{4.11}
$$

Furthermore, it follows from (1.5) that

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\leq C \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} \int_{|x_1-y_1|\leq\frac{|x_1|}{2}} \log\left(1+\frac{4x_2y_2}{|x_1-y_1|^2}\right)(|y_1|+y_2)^{-(m+2)}\,dy_1\,dy_2 \\
&\leq C \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} (|x_1|+y_2)^{-(m+2)}\,dy_2 \int_0^{\frac{|x_1|}{2}} \log\left(1+\frac{6x_2^2}{r^2}\right)dr \\
&\leq C \int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} (|x_1|+y_2)^{-(m+2)}\,dy_2 \left(\frac{|x_1|}{2}\log\left(1+\frac{24x_2^2}{|x_1|^2}\right)+\int_0^{\frac{|x_1|}{2}} \frac{12x_2^2}{r^2+6x_2^2}\,dr\right).
\end{aligned}
$$

Since $x \geq \log(1+x)$ for $x \geq 0$, we deduce in the case $N = 2$ that

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\frac{3x_2}{2}} \int_{D_1} G(x,y)f(y)^p\,dy \\
&\leq Cx_2^2|x_1|^{-1}(|x_1|+x_2)^{-(m+1)} + Cx_2|x_1|(|x_1|+x_2)^{-(m+2)} \leq Cx_2|x_1|^{-m-1}.
\end{aligned} \tag{4.12}
$$

On the other hand, in the case $N \geq 2$, by (2.4) and (2.5) with $\alpha = 1$ we have

$$
G(x,y) \leq C\frac{x_N y_N}{|x-y|^N} \tag{4.13}
$$

for all $y \in \mathbb{R}^N_+$. Since $|y'| \geq |x'|/2$ for $y' \in D_1$, by (4.13) we have

$$\begin{aligned}
&\int_{\frac{3x_N}{2}}^{\infty} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\leq C \int_{\frac{3x_N}{2}}^{\infty} \int_{|x'-y'|\leq \frac{|x'|}{2}} \frac{x_N y_N}{|x-y|^N} (|y'|+y_N)^{-(m+2)}\,dy'\,dy_N \\
&\leq C x_N \int_{\frac{3x_N}{2}}^{\infty} \int_{|x'-y'|\leq \frac{|x'|}{2}} y_N (|x'|+y_N)^{-(m+2)} (|x'-y'|+y_N-x_N)^{-N}\,dy'\,dy_N \\
&\leq C x_N \int_{\frac{3x_N}{2}}^{\infty} \int_0^{\infty} y_N (|x'|+y_N)^{-(m+2)} (r+y_N-x_N)^{-N} r^{N-2}\,dr\,dy_N.
\end{aligned}$$

So we obtain

$$\begin{aligned}
&\int_{\frac{3x_N}{2}}^{\infty} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \\
&\leq C x_N \int_{\frac{3x_N}{2}}^{\infty} \int_0^{\infty} y_N (|x'|+y_N)^{-(m+2)} (r+y_N-x_N)^{-2}\,dr\,dy_N \\
&\leq C x_N \int_{\frac{3x_N}{2}}^{\infty} (|x'|+y_N)^{-(m+2)} \frac{y_N}{y_N-x_N}\,dy_N \\
&\leq C x_N \int_0^{\infty} (|x'|+y_N)^{-(m+2)}\,dy_N \leq C x_N |x'|^{-m-1}.
\end{aligned}$$

This together with (4.9), (4.11) and (4.12) implies (4.8).

We deduce from (4.7) and (4.8) that

$$\int_0^{\infty} \int_{D_1} G(x,y)|y|^{-(m+2)}\,dy \leq \ C x_N \min\left\{|x'|^{-m-1}, x_N^{-m-1}\right\}.$$

This together with (3.14) implies that (4.3) with $k=1$.

Proof in the case $k=2$. Similarly to the case $k=1$, it suffices to consider the case $|x'| \neq 0$. We have

$$|x-y|^2 + 2x_N y_N \geq \sqrt{|x-y|^2+2x_N y_N}\sqrt{2x_N y_N} \geq |x'-y'|x_N$$

if $y_N \geq x_N/2$ and

$$|x-y|^2 + 2x_N y_N \geq |x-y|^2 \geq |x'-y'||x_N-y_N| \geq \frac{1}{2}|x'-y'|x_N$$

if $0 \leq y_N < x_N/2$. Then, in the case $N \geq 3$, it follows from (2.4) that

$$G(x,y) \leq C\frac{y_N}{|x'-y'|^{N-1}} \leq C\frac{y_N}{|x'|^{N-1}}, \qquad y' \in D_2.$$

Since $D_2 \subset \{y' \in \mathbb{R}^{N-1} : |y'| \le 3|x'|\}$, we have

$$
\begin{aligned}
&\int_{\frac{x_N}{2}}^{\infty}\int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le C|x'|^{-(N-1)}\int_{\frac{x_N}{2}}^{\infty}\int_{\frac{|x'|}{2}<|x'-y'|\le 2|x'|} y_N|y|^{-(m+2)}\,dy'\,dy_N \\
&\le C|x'|^{-(N-1)}\int_{\frac{x_N}{2}}^{\infty}\int_0^{3|x'|}(r+y_N)^{-(m+1)}r^{N-2}\,dr\,dy_N \\
&\le C\int_{\frac{x_N}{2}}^{\infty} y_N^{-(m+1)}\,dy_N \le Cx_N^{-m}.
\end{aligned}
\tag{4.14}
$$

In the case $N = 2$, it follows from (2.5) with $\alpha = 1/2$ that

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\infty}\int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_2^{\frac12}\int_{\frac{x_2}{2}}^{\infty}\int_{\frac{|x_1|}{2}<|x_1-y_1|\le 2|x_1|} y_2^{\frac12}|x_1-y_1|^{-1}|y|^{-(m+2)}\,dy_1\,dy_2 \\
&\le Cx_2^{\frac12}|x_1|^{-1}\int_{\frac{x_2}{2}}^{\infty}\int_0^{3|x_1|}(r+y_2)^{-m-\frac32}\,dr\,dy_2 \\
&\le Cx_2^{\frac12}\int_{\frac{x_2}{2}}^{\infty} y_2^{-m-\frac32}\,dy_2 \le Cx_2^{-m}.
\end{aligned}
\tag{4.15}
$$

On the other hand, in the case $N \ge 2$, by (3.1) and (4.6) we have

$$
\begin{aligned}
&\int_0^{\frac{x_N}{2}}\int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \\
&\le Cx_N^{-m}\int_0^{\frac{x_N}{2}}\int_{D_2}\frac{y_N}{|x'-y'|^{N-1-m}}|y|^{-(m+2)}\,dy'\,dy_N \\
&\le Cx_N^{-m}|x'|^{-(N-1)+m}\int_0^{3|x'|}\int_0^{\infty} r^{N-2}(r+y_N)^{-(m+1)}\,dy_N\,dr \\
&\le Cx_N^{-m}|x'|^{-(N-1)+m}\int_0^{3|x'|} r^{N-m-2}\,dr \\
&\le Cx_N^{-m}|x'|^{-(N-1)+m}|x'|^{N-1-m} \le Cx_N^{-m}.
\end{aligned}
$$

This together with (4.14) and (4.15) implies that

$$
\int_0^{\infty}\int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N^{-m}. \tag{4.16}
$$

Furthermore, it follows from (4.13) that

$$
G(x,y) \le C\frac{x_N y_N}{|x'-y'|^N} \le C\frac{x_N y_N}{|x'|^N}, \qquad y' \in D_2,
$$

and we obtain

$$\begin{aligned}
&\int_0^\infty \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy\\
&\le Cx_N|x'|^{-N}\int_0^\infty\int_0^{3|x'|} y_N(r+y_N)^{-(m+2)}r^{N-2}\,dr\,dy_N\\
&\le Cx_N|x'|^{-N}\int_0^\infty\int_0^{3|x'|} (r+y_N)^{-(m+1)}r^{N-2}\,dr\,dy_N\\
&\le Cx_N|x'|^{-N}\int_0^{3|x'|} r^{N-m-2}\,dr \le Cx_N|x'|^{-m-1-\varepsilon}\int_0^{3|x'|}|x'|^{-N+m+1+\varepsilon}r^{N-m-2}\,dr\\
&\le Cx_N|x'|^{-m-1-\varepsilon}\int_0^{3|x'|} r^{-1+\varepsilon}\,dr \le Cx_N|x'|^{-m-1},
\end{aligned}$$

where ε is as in (3.9). This together with (4.16) yields

$$\int_0^\infty \int_{D_2} G(x,y)|y|^{-(m+2)}\,dy \le Cx_N \min\left\{|x'|^{-m-1}, x_N^{-m-1}\right\}.$$

Therefore, by (3.14) we obtain (4.3) with $k=2$.

Proof in the case $k=3$. Since

$$|y'| \ge |x'-y'|-|x'|>0, \qquad y\in D_3, \tag{4.17}$$

by (4.13) we have

$$\begin{aligned}
&\int_0^\infty \int_{D_3} G(x,y)|y|^{-(m+2)}\,dy\\
&\le Cx_N\int_0^\infty\int_{|x'-y'|>2|x'|} y_N|x'-y'|^{-N}(|y'-x'|-|x'|+y_N)^{-(m+2)}\,dy'\,dy_N\\
&\le Cx_N\int_0^\infty\int_{2|x'|}^\infty r^{-2}(r-|x'|+y_N)^{-(m+1)}\,dr\,dy_N\\
&\le Cx_N\int_0^\infty (|x'|+y_N)^{-(m+1)}\left(\int_{2|x'|}^\infty r^{-2}\,dr\right)dy_N \le Cx_N|x'|^{-m-1}.
\end{aligned}\tag{4.18}$$

It follows from (2.4), (2.5) and (4.17) that

$$\begin{aligned}
&\int_{\frac{x_N}{2}}^\infty \int_{D_3} G(x,y)|y|^{-(m+2)}\,dy\\
&\le C\int_{\frac{x_N}{2}}^\infty\int_{|x'-y'|>2|x'|} |x'-y'|^{-(N-2)}(|y'-x'|-|x'|+y_N)^{-(m+2)}\,dy'\,dy_N\\
&\le C\int_{\frac{x_N}{2}}^\infty\int_{2|x'|}^\infty (r-|x'|+y_N)^{-(m+2)}\,dr\,dy_N\\
&\le C\int_{\frac{x_N}{2}}^\infty (|x'|+y_N)^{-(m+1)}\,dy_N \le C|x|^{-m} \le Cx_N^{-m} \quad \text{if} \quad N\ge 3,
\end{aligned}\tag{4.19}$$

and

$$
\begin{aligned}
&\int_{\frac{x_2}{2}}^{\infty}\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy\\
&\le C\int_{\frac{x_2}{2}}^{\infty}\int_{|x_1-y_1|>2|x_1|}\left(\frac{x_2y_2}{|x_2-y_2|^2}\right)^{\alpha}(|y_1-x_1|-|x_1|+y_2)^{-(m+2)}\,dy_1\,dy_2\\
&\le C\int_{\frac{x_2}{2}}^{\infty}\int_{2|x_1|}^{\infty}\left(\frac{x_2}{|x_2-y_2|^2}\right)^{\alpha}(r-|x_1|+y_2)^{-(m+2)+\alpha}\,dr\,dy_2\\
&\le C\biggl(\int_{\frac{x_2}{2}}^{2x_2}+\int_{2x_2}^{\infty}\biggr)\left(\frac{x_2}{|x_2-y_2|^2}\right)^{\alpha}(|x_1|+y_2)^{-(m+1)+\alpha}\,dy_2\\
&\le Cx_2^{\alpha}|x|^{-(m+1)+\alpha}\int_{\frac{x_2}{2}}^{2x_2}|x_2-y_2|^{-2\alpha}\,dy_2+Cx_2^{-\alpha}\int_{2x_2}^{\infty}(|x_1|+y_2)^{-(m+1)+\alpha}\,dy_2\\
&\le Cx_2^{1-\alpha}|x|^{-(m+1)+\alpha}+Cx_2^{-\alpha}|x|^{-m+\alpha}\le Cx_2^{-m}\quad\text{if}\quad N=2,
\end{aligned}
\tag{4.20}
$$

where $\alpha\in(0,1/2)$ with $\alpha<m$.

On the other hand, it follows from (2.4) and (2.5) with $\alpha=1$ that

$$
G(x,y)\le C\frac{x_Ny_N}{|x_N-y_N|^{1+m+\varepsilon}|x'-y'|^{N-1-m-\varepsilon}}\le Cx_N^{-m-\varepsilon}\frac{y_N}{|x'-y'|^{N-1-m-\varepsilon}}
$$

for all $y\in\mathbb{R}^N_+$ with $0\le y_N\le x_N/2$, where ε is as (3.9). This together with (4.17) yields

$$
\begin{aligned}
&\int_0^{\frac{x_N}{2}}\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy'\,dy_N\\
&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}\int_{|x'-y'|>2|x'|}\frac{y_N}{|x'-y'|^{N-1-m-\varepsilon}}(|x'-y'|-|x'|+y_N)^{-(m+2)}\,dy'\\
&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}\int_{2|x'|}^{\infty}y_Nr^{-1+m+\varepsilon}(r-|x'|+y_N)^{-(m+2)}\,dr\,dy_N\\
&\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}\biggl(\int_{2|x'|}^{2|x'|+y_N}+\int_{2|x'|+y_N}^{\infty}\biggr)y_Nr^{-1+\varepsilon}(r-|x'|+y_N)^{-2}\,dr\,dy_N\\
&=: Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}(I_1(y_N)+I_2(y_N))\,dy_N.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
I_1(y_N)&\le C(|x'|+y_N)^{-1}\int_{2|x'|}^{2|x'|+y_N}r^{-1+\varepsilon}\,dr\le C(|x'|+y_N)^{-1+\varepsilon},\\
I_2(y_N)&\le C\int_{2|x'|+y_N}^{\infty}(r-|x'|+y_N)^{-2+\varepsilon}\,dr\le C(|x'|+y_N)^{-1+\varepsilon}.
\end{aligned}
$$

These imply that

$$\begin{aligned}(4.21)\qquad &\int_0^{\frac{x_N}{2}}\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy'\,dy_N \\ &\le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}}(|x'|+y_N)^{-1+\varepsilon}\,dy_N \le Cx_N^{-m-\varepsilon}\int_0^{\frac{x_N}{2}} y_N^{-1+\varepsilon}\,dy_N \le Cx_N^{-m}.\end{aligned}$$

Therefore, by (4.19), (4.20) and (4.21) we obtain

$$\int_0^\infty\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy_1\,dy_2 \le Cx_N^{-m}. \tag{4.22}$$

We deduce from (3.14), (4.18) and (4.22) that

$$\int_0^\infty\int_{D_3} G(x,y)|y|^{-(m+2)}\,dy_1\,dy_2 \le Cx_N\min\{|x'|^{-m-1}, x_N^{-m-1}\} \le Cx_N|x|^{-m-1},$$

which implies (4.3) with $k=3$. Thus (4.3) holds for $k=1,2,3$, and the proof of Lemma 4.1 is complete. □

Lemma 4.2. *Let $p>p_*$. Assume that*

$$0\le f(x)\le\begin{cases}|x|^{-m}, & x\in D_{\rm in},\\ (1+x_N)|x|^{-N}, & x\in D_{\rm out}.\end{cases} \tag{4.23}$$

Then there exists a positive constant C such that

$$[(-\Delta_D)^{-1}f^p](x)\le Cx_N|x|^{-N},\qquad x\in D_{\rm out}.$$

Proof. Let $x=(x',x_N)\in D_{\rm out}$. We divide $\mathbb{R}^N_+$ into the following seven sets:

$$\begin{aligned}
D_1 &:= \left\{y\in\mathbb{R}^N_+ : |y'|\le\frac14,\quad y_N\le\frac{1+x_N}{4}\right\},\\
D_2 &:= \left\{y\in\mathbb{R}^N_+ : |y'|\le\frac14,\quad |y'|\le\frac{|x'|}{2},\quad y_N\ge\frac{1+x_N}{4}\right\},\\
D_3 &:= \left\{y\in\mathbb{R}^N_+ : |y'|\le\frac14,\quad \frac{|x'|}{2}\le|y'|\le\frac{3|x'|}{2},\quad y_N\ge\frac{1+x_N}{4}\right\},\\
D_4 &:= \left\{y\in\mathbb{R}^N_+ : |y'|\le\frac14,\quad |y'|\ge\frac{3|x'|}{2},\quad y_N\ge\frac{1+x_N}{4}\right\},\\
D_5 &:= \left\{y\in\mathbb{R}^N_+ : \frac14\le|y'|\le\frac{|x'|}{2}\right\},\\
D_6 &:= \left\{y\in\mathbb{R}^N_+ : |y'|\ge\frac14,\quad \frac{|x'|}{2}\le|y'|\le\frac{3|x'|}{2}+1+x_N,\right\},\\
D_7 &:= \left\{y\in\mathbb{R}^N_+ : |y'|\ge\frac{3|x'|}{2}+1+x_N\right\}.
\end{aligned}$$

Then, by (4.1) it suffices to prove that

$$\int_{D_k} G(x,y)f(y)^p\,dy\le Cx_N|x|^{-N},\quad k=1,\dots,7. \tag{4.24}$$

Here we recall that $mp=m+2$.

Proof in the case $k = 1$. By (3.1) and (4.23) we have

$$0 \le f(x) \le C|x|^{-m}, \qquad x \in \mathbb{R}^N_+. \tag{4.25}$$

For any $y \in D_1$, since

$$|x - y| \ge |x| - |y| \ge |x| - \left(\frac{1}{2} + \frac{x_N}{4}\right) \ge \frac{3|x|}{4} - \frac{1}{2} \ge \frac{|x|}{4},$$

by (4.13) we see that

$$G(x, y) \le C\frac{x_N y_N}{|x - y|^N} \le Cx_N|x|^{-N}y_N, \qquad y \in D_1. \tag{4.26}$$

Then it follows from (3.1), (4.25) and (4.26) that

$$\begin{aligned}
\int_{D_1} G(x,y)f(y)^p\,dy &\le Cx_N|x|^{-N}\int_{D_1} y_N|y|^{-(m+2)}\,dy \\
&\le Cx_N|x|^{-N}\int_{|y'|\le\frac{1}{4}}\int_0^{\frac{1+x_N}{4}} (|y'| + y_N)^{-(m+1)}\,dy_N\,dy' \\
&\le Cx_N|x|^{-N}\int_{|y'|\le\frac{1}{4}} |y'|^{-m}\,dy' \le Cx_N|x|^{-N}\int_0^{\frac{1}{4}} r^{-m+N-2}\,dr \le Cx_N|x|^{-N}.
\end{aligned}$$

This implies (4.24) with $k = 1$.

Proof in the case $k = 2$. For any $y \in D_2$, since

$$|x' - y'| \ge |x'| - |y'| \ge \frac{|x'|}{2},$$

by (4.13) we have

$$G(x, y) \le C\frac{x_N y_N}{|x - y|^N} \le C\frac{x_N y_N}{|x' - y'|^N} \le Cx_N y_N|x'|^{-N}, \qquad y \in D_2. \tag{4.27}$$

On the other hand, by (3.1) and (4.23) we have

$$0 \le f(y) \le C(1 + y_N)|y|^{-N} \le Cy_N|y|^{-N}, \qquad y \in D_2 \cup D_3 \cup D_4. \tag{4.28}$$

Furthermore, it follows from $p > p_*$ that

$$-Np + p + N + 1 < 0. \tag{4.29}$$

Then, by (4.27), (4.28) and (4.29) we obtain

$$\begin{aligned}
\int_{D_2} G(x,y)f(y)^p\,dy &\le Cx_N|x'|^{-N}\int_{D_2} y_N^{p+1}|y|^{-Np}\,dy \\
&\le Cx_N|x'|^{-N}\int_{|y'|\le\frac{|x'|}{2}}\int_{\frac{1+x_N}{4}}^{\infty} (|y'| + y_N)^{-Np+p+1}\,dy_N\,dy' \\
&\le Cx_N|x'|^{-N}\int_{|y'|\le\frac{|x'|}{2}} (1 + x_N + |y'|)^{-Np+p+2}\,dy' \\
&\le Cx_N|x'|^{-N}\int_0^{\frac{|x'|}{2}} (1 + r)^{-Np+p+N}\,dr \le Cx_N|x'|^{-N}.
\end{aligned} \tag{4.30}$$

On the other hand, in the case $N \ge 3$, by (2.4) we see that

$$G(x, y) \le C|x' - y'|^{-(N-2)} \le C|x'|^{-(N-2)}, \qquad y \in D_2.$$

Then it follows from (4.28) and (4.29) that

$$
\begin{aligned}
\int_{D_2} G(x,y)f(y)^p\,dy &\le C|x'|^{-(N-2)}\int_{D_2} y_N^p|y|^{-Np}\,dy\\
&\le C|x'|^{-(N-2)}\int_{|y'|\le\frac{|x'|}{2}}\int_{\frac{1+x_N}{4}}^{\infty}(|y'|+y_N)^{-Np+p}\,dy_N\,dy'\\
&\le C|x'|^{-(N-2)}\int_{|y'|\le\frac{|x'|}{2}}(1+|y'|+x_N)^{-Np+p+1}\,dy'\\
&\le C|x'|^{-(N-2)}\int_0^{\frac{|x'|}{2}} r^{N-2}(1+r+x_N)^{-Np+p+1}\,dr\\
&\le C(1+x_N)^{-Np+p+2}\le C(1+x_N)^{-(N-1)}\le Cx_N^{-(N-1)}.
\end{aligned}
\tag{4.31}
$$

Furthermore, in the case $N=2$, since $p>3$, by (2.5) with $\alpha\in(0,1/2)$ and (4.28) we obtain

$$
\begin{aligned}
\int_{D_2} G(x,y)f(y)^p\,dy &\le C\int_{D_2}\left(\frac{x_2y_2}{|x_2-y_2|^2}\right)^{\alpha} y_2^p|y|^{-2p}\,dy\\
&\le Cx_2^{\alpha}\int_{\frac{1+x_2}{4}}^{\infty}\int_0^{\frac{|x_1|}{2}}|x_2-y_2|^{-2\alpha}(r+y_2)^{-p+\alpha}\,dr\,dy_2\\
&\le Cx_2^{\alpha}\left(\int_{\frac{1+x_2}{4}}^{1+2x_2}+\int_{1+2x_2}^{\infty}\right)|x_2-y_2|^{-2\alpha}y_2^{-p+\alpha+1}\,dy_2\\
&\le Cx_2^{\alpha}\Big((1+x_2)^{1-2\alpha}(1+x_2)^{-p+\alpha+1}+(1+x_2)^{-2\alpha}(1+x_2)^{-p+\alpha+2}\Big)\\
&\le C(1+x_2)^{-p+2}\le C(1+x_2)^{-1}\le Cx_2^{-1}.
\end{aligned}
$$

This together with (3.14), (4.30) and (4.31) yields

$$
\int_{D_2} G(x,y)f(y)^p\,dy\le Cx_N\min\{|x'|^{-N},x_N^{-N}\}\le Cx_N|x|^{-N}.
$$

Thus (4.24) holds for $k=2$.

Proof in the case $k=3$. In the case $N\ge3$, by (2.4) we have

$$
G(x,y)\le C\frac{x_Ny_N}{|x'-y'|^{N-2}(x_N+y_N)^2},\qquad y\in\mathbb{R}^N_+. \tag{4.32}
$$

Then, since $|x'-y'|\le C|x'|$ for any $y\in D_3$, by (4.28) and (4.29) we see that

$$
\begin{aligned}
&\int_{D_3} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_{\frac{1+x_N}{4}}^{\infty}\int_{\frac{|x'|}{2}\le|y'|\le\frac{3|x'|}{2}}\frac{y_N^{p+1}}{|x'-y'|^{N-2}(x_N+y_N)^2}|y|^{-Np}\,dy'\,dy_N\\
&\le Cx_N\int_{\frac{1+x_N}{4}}^{\infty}\int_{|x'-y'|\le C|x'|} y_N^{p-1}(|x'|+y_N)^{-Np}|x'-y'|^{N-2}\,dy'\,dy_N\\
&\le Cx_N\int_{\frac{1+x_N}{4}}^{\infty}\int_0^{C|x'|} y_N^{p-1}(|x'|+y_N)^{-Np}r^{-(N-2)}r^{N-2}\,dr\,dy_N\\
&\le Cx_N\int_{\frac{1+x_N}{4}}^{\infty} y_N^{p-1}(|x'|+y_N)^{-Np+1}\,dy_N\le Cx_N\int_{\frac{1+x_N}{4}}^{\infty}(|x'|+y_N)^{-Np+p}\,dy_N\\
&\le Cx_N(1+|x|)^{-Np+p+1}\le Cx_N|x|^{-N}.
\end{aligned}
\tag{4.33}
$$

In the case $N=2$, for any $\alpha\in(0,1]$, by (2.5) we obtain

$$
G(x,y)\le C\left(\frac{x_2y_2}{|x_1-y_1||x_2-y_2|}\right)^{\alpha},\qquad y\in\mathbb{R}^2_+.
$$

For any $y\in D_3$, since $x\in D_{\text{out}}$ and $|x_1|/2\le|y_1|\le 1/4$, we see that $x_2\ge 1-|x_1|\ge 1/2$. This implies that $x_2^\alpha\le 2x_2$ for any $\alpha\in(0,1)$ and $x_2\ge(1+x_2)/4$. Then, since $p>3$, by (4.28) we have

$$
\begin{aligned}
&\int_{D_3} G(x,y)f(y)^p\,dy\\
&\le Cx_2^\alpha\int_{\frac{1+x_2}{4}}^{\infty}\int_{\frac{|x_1|}{2}\le|y_1|\le\frac{3|x_1|}{2}}\left(\frac{y_2}{|x_1-y_1||x_2-y_2|}\right)^{\alpha}y_2^p|y|^{-2p}\,dy_1\,dy_2\\
&\le Cx_2^\alpha\int_{\frac{1+x_2}{4}}^{\infty}\int_{|x_1-y_1|\le C|x_1|} y_2^{p+\alpha}|x_2-y_2|^{-\alpha}(|x_1|+y_2)^{-2p}|x_1-y_1|^{-\alpha}\,dy_1\,dy_2\\
&\le Cx_2^\alpha\left(\int_{\frac{1+x_2}{4}}^{2x_2}+\int_{2x_2}^{\infty}\right)y_2^\alpha|x_2-y_2|^{-\alpha}(|x_1|+y_2)^{-p+1-\alpha}\,dy_2\\
&\le Cx_2^\alpha\left(|x|^{-p+1}\int_{\frac{1+x_2}{4}}^{2x_2}|x_2-y_2|^{-\alpha}\,dy_2+\int_{2x_2}^{\infty}(|x_1|+y_2)^{-p+1-\alpha}\,dy_2\right)\\
&\le Cx_2|x|^{-2}\left(|x|^{-p+3}+|x|^{-p+4-\alpha}\right)\le Cx_2|x|^{-2},
\end{aligned}
\tag{4.34}
$$

where $\alpha\in(0,1)$ with $-p+4<\alpha$. This together with (4.33) implies that (4.24) with $k=3$.

Proof in the case $k=4$. For any $y\in D_4$, since

$$
|x'-y'|\ge|y'|-|x'|\ge\frac{|x'|}{2},
$$

by (4.13) we obtain

$$G(x,y) \le C\frac{x_N y_N}{|x'-y'|^N} \le C x_N y_N |x'|^{-N}. \tag{4.35}$$

Then it follows from (4.28) and (4.29) that

$$\begin{aligned}
\int_{D_4} G(x,y)f(y)^p\,dy &\le C x_N |x'|^{-N} \int_{\frac{1+x_N}{4}}^{\infty} \int_{\frac{3|x'|}{2}\le |y'|\le \frac14} y_N^{p+1}|y|^{-Np}\,dy'\,dy_N \\
&\le C x_N |x'|^{-N} \int_{\frac14}^{\infty}\int_0^{\frac14} y_N^{p+1}(r+y_N)^{-Np+N-2}\,dr\,dy_N \\
&\le C x_N |x'|^{-N} \int_{\frac14}^{\infty} y_N^{-Np+p+N}\,dy_N \le C x_N |x'|^{-N}.
\end{aligned} \tag{4.36}$$

For any $y\in D_4$, since $|x'|\le 1/6$, we see that

$$\left\{y'\in\mathbb{R}^{N-1} : \frac{3|x'|}{2}\le |y'|\le \frac14\right\} \subset \left\{y'\in\mathbb{R}^{N-1} : |x'-y'|\le 1\right\}. \tag{4.37}$$

Then, in the case $N\ge 3$, by (4.28), (4.29), (4.32) and (4.37) we have

$$\begin{aligned}
&\int_{D_4} G(x,y)f(y)^p\,dy \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} \int_{\frac{3|x'|}{2}\le |y'|\le \frac14} \frac{y_N^{p+1}}{|x'-y'|^{N-2}(x_N+y_N)^2}|y|^{-Np}\,dy'\,dy_N \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} \int_{|x'-y'|\le 1} (|x'|+y_N)^{-Np+p-1}|x'-y'|^{-N+2}\,dy'\,dy_N \\
&\le C x_N \int_{\frac{1+x_N}{4}}^{\infty} (|x'|+y_N)^{-Np+p-1}\,dy_N \le C x_N\left(|x'|+\frac{1+x_N}{4}\right)^{-Np+p} \\
&\le C x_N\left(|x'|+\frac{1+x_N}{4}\right)^{-N} \le C x_N^{-(N-1)}.
\end{aligned} \tag{4.38}$$

On the other hand, in the case $N=2$, for $y\in D_4$, we have $|x_1|\le 1/6$ and it follows from $x\in D_{\rm out}$ that $x_2\ge 1-|x_1|\ge 5/6$ and $2x_2\ge (1+x_2)/4$. Then, similarly to (4.34), by (4.37) we have

$$
\begin{aligned}
&\int_{D_4} G(x,y) f(y)^p \, dy \\
&\le C x_2^{\alpha} \int_{\frac{1+x_2}{4}}^{\infty} \int_{\frac{3|x_1|}{2} \le |y_1| \le \frac{1}{4}} \left(\frac{y_2}{|x_1 - y_1||x_2 - y_2|} \right)^{\alpha} y_2^p |y|^{-2p} \, dy_1 \, dy_2 \\
&\le C x_2^{\alpha} \int_{\frac{1+x_2}{4}}^{\infty} \int_{|x_1 - y_1| \le 1} y_2^{-p+\alpha} |x_2 - y_2|^{-\alpha} |x_1 - y_1|^{-\alpha} \, dy_1 \, dy_2 \\
&\le C x_2^{\alpha} \left(\int_{\frac{1+x_2}{4}}^{2x_2} + \int_{2x_2}^{\infty} \right) y_2^{-p+\alpha} |x_2 - y_2|^{-\alpha} \, dy_2 \\
&\le C x_2^{\alpha} \left(x_2^{-p+\alpha} \int_{\frac{x_2}{4}}^{2x_2} |x_2 - y_2|^{-\alpha} \, dy_2 + \int_{2x_2}^{\infty} y_2^{-p} \, dy_2 \right) \\
&\le C x_2^{-p+1+\alpha} \le C x_2^{-1},
\end{aligned}
$$

where $\alpha \in (0,1)$. This together with (3.14), (4.36) and (4.38) implies that

$$
\int_{D_4} G(x,y) f(y)^p \, dy \le C x_N \min\{|x'|^{-N}, x_N^{-N}\} \le C x_N |x|^{-N}.
$$

Thus (4.24) holds for $k = 4$.

Proof in the case $k = 5$. Since $x \in D_{\text{out}}$, we have

$$
\begin{aligned}
|x - y| &\ge |x| - (|y'| + y_N) \ge |x| - \frac{|x'|}{2} - \frac{1 + x_N}{4} \\
&\ge |x| - \frac{|x'|}{4} - \frac{\sqrt{2}|x|}{4} - \frac{1}{4} \ge \frac{3 - \sqrt{2}}{4}|x| - \frac{1}{4} \ge \frac{2 - \sqrt{2}}{4}|x|
\end{aligned}
$$

for $y \in D_5$ with $0 \le y_N \le (1 + x_N)/4$. Then it follows from (4.13) that

$$
G(x,y) \le C \frac{x_N y_N}{|x - y|^N} \le C x_N y_N |x|^{-N}
$$

for $y \in D_5$ with $0 \le y_N \le (1 + x_N)/4$. By (4.23) and (4.29) we obtain

$$
\begin{aligned}
&\int_{D_5 \cap \{0 \le y_N \le \frac{1+x_N}{4}\}} G(x,y) f(y)^p \, dy \\
&\le C x_N |x|^{-N} \int_0^{\frac{1+x_N}{4}} \int_{\frac{1}{4} \le |y'| \le \frac{|x'|}{2}} y_N (1 + y_N)^p |y|^{-Np} \, dy' \, dy_N \\
&\le C x_N |x|^{-N} \int_0^{\frac{1+x_N}{4}} \int_{\frac{1}{4} \le r \le \frac{|x'|}{2}} (1 + y_N)^{p+1} (y_N + r)^{-Np+N-2} \, dr \, dy_N \\
&\le C x_N |x|^{-N} \int_0^{\infty} (1 + y_N)^{-Np+N+p} \, dy_N \le C x_N |x|^{-N}.
\end{aligned} \tag{4.39}
$$

On the other hand, since $|x'-y'|\ge |x'|/2$ for any $y\in D_5$, similarly to (4.36), by (4.23), (4.29) and (4.35) we obtain

$$
\begin{aligned}
&\int_{D_5\cap\{y_N\ge\frac{1+x_N}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N|x'|^{-N}\int_{\frac{1+x_N}{4}}^{\infty}\int_{\frac14\le|y'|\le\frac{|x'|}{2}} y_N(1+y_N)^p|y|^{-Np}\,dy'\,dy_N\\
&\le Cx_N|x'|^{-N}\int_{\frac14}^{\infty}\int_{\frac14\le r\le\frac{|x'|}{2}} y_N^{p+1}(r+y_N)^{-Np+N-2}\,dr\,dy_N\\
&\le Cx_N|x'|^{-N}\int_{\frac14}^{\infty} y_N^{-Np+N+p}\,dy'\,dy_N\le Cx_N|x'|^{-N}.
\end{aligned}
\tag{4.40}
$$

Furthermore, in the case $N\ge 3$, by (4.23), (4.29) and (4.32) we have

$$
\begin{aligned}
&\int_{D_5\cap\{y_N\ge\frac{1+x_N}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_{\frac{1+x_N}{4}}^{\infty}\int_{\frac14\le|y'|\le\frac{|x'|}{2}} \frac{y_N}{|x'-y'|^{N-2}(x_N+y_N)^2}(1+y_N)^p|y|^{-Np}\,dy'\,dy_N\\
&\le Cx_N(1+x_N)^{-1}\int_{\frac{1+x_N}{4}}^{\infty}\int_{\frac14}^{\frac{|x'|}{2}}(r+y_N)^{-Np}(1+y_N)^p\,dr\,dy_N\\
&\le Cx_N(1+x_N)^{-1}\int_{\frac{1+x_N}{4}}^{\infty}(1+y_N)^{-Np+p+1}\,dy_N\\
&\le Cx_N(1+x_N)^{-Np+p+1}\le Cx_N(1+x_N)^{-N}\le Cx_N^{-(N-1)}.
\end{aligned}
\tag{4.41}
$$

In the case where $N=2$ and $x_2\ge 1/7$, since $2x_2\ge(1+x_2)/4$ and $p>3$, it follows from (2.5) and (4.23) that

$$
\begin{aligned}
&\int_{D_5\cap\{y_2\ge\frac{1+x_2}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_2^{\alpha}\int_{\frac{1+x_2}{4}}^{\infty}\int_{\frac14\le|y_1|\le\frac{|x_1|}{2}}\left(\frac{y_2}{|x_2-y_2|^2}\right)^{\alpha}(1+y_2)^p|y|^{-2p}\,dy_1\,dy_2\\
&\le Cx_2^{\alpha}\int_{\frac{1+x_2}{4}}^{\infty}\int_{\frac14}^{\frac{|x_1|}{2}} y_2^{p+\alpha}|x_2-y_2|^{-2\alpha}(r+y_2)^{-2p}\,dr\,dy_2\\
&\le Cx_2^{\alpha}\left(\int_{\frac{1+x_2}{4}}^{2x_2}+\int_{2x_2}^{\infty}\right)y_2^{-p+\alpha+1}|x_2-y_2|^{-2\alpha}\,dy_2\le Cx_2^{-p+2}\le Cx_2^{-1},
\end{aligned}
\tag{4.42}
$$

where $\alpha \in (0,1/2)$. In the case where $N=2$ and $x_2 \le 1/7$, since $2x_2 \le (1+x_2)/4$ and $p>3$, by (4.13) and (4.23) we have

$$\begin{aligned}
&\int_{D_5\cap\{y_2\ge\frac{1+x_2}{4}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_2\int_{\frac{1+x_2}{4}}^{\infty}\int_{\frac14\le|y_1|\le\frac{|x_1|}{2}} \frac{y_2(1+y_2)^p}{|x_2-y_2|^2}|y|^{-2p}\,dy_1\,dy_2\\
&\le Cx_2\int_{2x_2}^{\infty}\int_{\frac14}^{\infty}(y_2-x_2)^{-2}(r+y_2)^{-p+1}\,dr\,dy_2\\
&\le Cx_2^{-1}\int_{2x_2}^{\infty}(1+y_2)^{-p+2}\,dy_2 \le Cx_2^{-1}(1+x_2)^{-p+3}\le Cx_2^{-1}.
\end{aligned}$$

This together with (3.14), (4.40), (4.41) and (4.42) yields

$$\int_{D_5\cap\{y_N\ge\frac{1+x_N}{4}\}} G(x,y)f(y)^p\,dy \le Cx_N\min\{|x'|^{-N},x_N^{-N}\}\le Cx_N|x|^{-N}. \tag{4.43}$$

Therefore, by (4.39) and (4.43) we obtain (4.24) with $k=5$.

Proof in the case $k=6$. We recall that

$$D_6\subset\left\{y\in\mathbb{R}^N_+ : 0\le|x'-y'|\le\frac{5|x'|}{2}+1+x_N\right\}.$$

Put

$$A=\max\left\{\frac14,\frac{|x'|}{2}\right\}\ge\frac14.$$

Then $2A\ge|x'|$. In the case $N\ge3$, by (2.4) we have

$$G(x,y)\le C\frac{x_Ny_N}{|x'-y'|^{N-2}(|x'-y'|+x_N+y_N)^2},\qquad y\in\mathbb{R}^N_+.$$

It follows from (4.23) and (4.29) that

$$\begin{aligned}
&\int_{D_6\cap\{0\le y_N\le\frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_0^{\frac{x_N}{2}}\int_{A\le|y'|\le\frac{3|x'|}{2}+1+x_N}\frac{y_N(1+y_N)^p|y|^{-Np}}{|x'-y'|^{N-2}(|x'-y'|+x_N+y_N)^2}\,dy'\,dy_N\\
&\le Cx_N\int_0^{\frac{x_N}{2}}\int_0^{\frac{5|x'|}{2}+1+x_N} y_N(A+y_N)^{-Np+p}(r+y_N)^{-2}\,dr\,dy_N\\
&\le Cx_N\int_0^{\frac{x_N}{2}}(A+y_N)^{-Np+p}\,dy_N\le Cx_NA^{-Np+p+1}\le Cx_NA^{-N}\le Cx_N|x'|^{-N}.
\end{aligned} \tag{4.44}$$

In the case $N=2$, since $p>3$, by (4.10) and (4.23) we have

$$
\begin{aligned}
&\int_{D_6\cap\{0\le y_2\le \frac{x_2}{2}\}} G(x,y)f(y)^p\,dy\\
&\le C\int_0^{\frac{x_2}{2}}\int_{A\le |y_1|\le \frac{3|x_1|}{2}+1+x_2}(1+y_2)^p|y|^{-2p}\,dy_1\,dy_2 \qquad (4.45)\\
&\le C\int_0^{\frac{x_2}{2}}(A+y_2)^{-p+1}\,dy_2\le Cx_2A^{-p+1}\le Cx_2A^{-2}\le Cx_2|x_1|^{-2}.
\end{aligned}
$$

Furthermore, by (4.13) we see that

$$
G(x,y)\le C\frac{x_Ny_N}{|x_N-y_N|^N},\qquad y\in\mathbb{R}^N_+.
$$

Then it follows from (4.23) and (4.29) that

$$
\begin{aligned}
&\int_{D_6\cap\{0\le y_N\le \frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_0^{\frac{x_N}{2}}\int_{A\le |y'|\le \frac{3|x'|}{2}+1+x_N} y_N(1+y_N)^p|y|^{-Np}|x_N-y_N|^{-N}\,dy'\,dy_N\\
&\le Cx_N^{-N+1}\int_0^{\frac{x_N}{2}}\int_{A\le |y'|\le \frac{3|x'|}{2}+1+x_N} y_N(1+y_N)^p(r+y_N)^{-Np+N-2}\,dr\,dy_N\\
&\le Cx_N^{-N+1}\int_0^{\frac{x_N}{2}}(A+y_N)^{-Np+p+N}\,dy_N\le Cx_N^{-N+1}.
\end{aligned}
$$

This together with (3.14), (4.44) and (4.45) implies that

$$
\int_{D_6\cap\{0\le y_N\le \frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\le Cx_N\min\{|x'|^{-N},x_N^{-N}\}\le Cx_N|x|^{-N}. \qquad (4.46)
$$

On the other hand, in the case $N\ge 3$, similarly to (4.44), we have

$$
\begin{aligned}
&\int_{D_6\cap\{y_N\ge \frac{x_N}{2}\}} G(x,y)f(y)^p\,dy\\
&\le Cx_N\int_{\frac{x_N}{2}}^\infty\int_{A\le |y'|\le \frac{3|x'|}{2}+1+x_N}\frac{y_N(1+y_N)^p|y|^{-Np}}{|x'-y'|^{N-2}(|x'-y'|+x_N+y_N)^2}\,dy'\,dy_N \qquad (4.47)\\
&\le Cx_N\int_{\frac{x_N}{2}}^\infty(A+y_N)^{-Np+p}\,dy_N\le Cx_N(A+x_N)^{-Np+p+1}\le Cx_N|x|^{-N}.
\end{aligned}
$$

Furthermore, in the case $N = 2$, since $p > 3$, by (2.5) and (4.23) we obtain

$$\begin{aligned}
&\int_{D_6\cap\{\frac{x_2}{2}\le y_2\le 2x_2\}} G(x,y)f(y)^p\,dy \\
&\le Cx_2^\alpha \int_{\frac{x_2}{2}}^{2x_2}\int_{A\le|y_1|\le\frac{|x_1|}{3}+1+x_2}\left(\frac{y_2}{|x_2-y_2|^2}\right)^\alpha (1+y_2)^p|y|^{-2p}\,dy_1\,dy_2 \\
&\le Cx_2^\alpha \int_{\frac{x_2}{2}}^{2x_2} y_2^\alpha|x_2-y_2|^{-2\alpha}(A+y_2)^{-p+1}\,dy_2 \\
&\le Cx_2\left(A+\frac{x_2}{2}\right)^{-p+1} \le Cx_2\left(A+\frac{x_2}{2}\right)^{-2} \le Cx_2|x|^{-2},
\end{aligned} \tag{4.48}$$

where $\alpha \in (0, 1/2)$. In addition, since it follows from (4.13) that

$$G(x,y) \le C\frac{x_2y_2}{|x-y|^2} \le C\frac{x_2y_2}{(|x_1-y_1|+y_2)^2}$$

for any $y_2 \ge 2x_2$, by (4.23) we see that

$$\begin{aligned}
&\int_{D_6\cap\{y_2\ge 2x_2\}} G(x,y)f(y)^p\,dy \\
&\le Cx_2\int_{2x_2}^\infty\int_{A\le|y_1|\le\frac{|x_1|}{3}+1+x_2}\frac{y_2}{(|x_1-y_1|+y_2)^2}(1+y_2)^p|y|^{-2p}\,dy_1\,dy_2 \\
&\le Cx_2\int_{2x_2}^\infty\int_0^{\frac{5|x_1|}{2}+1+x_2} y_2(r+y_2)^{-2}(A+y_2)^{-p}\,dr\,dy_2 \\
&\le Cx_2\int_{2x_2}^\infty (A+y_2)^{-p}\,dy_2 \le Cx_2(A+2x_2)^{-p+1} \le Cx_2(A+2x_2)^{-2} \le Cx_2|x|^{-2}.
\end{aligned}$$

This together with (4.47) and (4.48) implies that

$$\int_{D_6\cap\{y_N\ge\frac{x_N}{2}\}} G(x,y)f(y)^p\,dy \le Cx_N|x|^{-N}. \tag{4.49}$$

Combining (4.46) and (4.49), we obtain (4.24) with $k = 6$.

Proof in the case $k = 7$. For any $y \in D_7$, by (4.13) we have

$$G(x,y) \le C\frac{x_Ny_N}{|x'-y'|^N} \le Cx_Ny_N|x|^{-N}.$$

Since $x \in D_{\text{out}}$, by (4.23) and (4.29) we see that

$$\begin{aligned}
&\int_{D_7} G(x,y)f(y)^p\,dy \\
&\le Cx_N|x|^{-N}\int_0^\infty\int_{|y'|\ge\frac{3|x'|}{2}+1+x_N} y_N(1+y_N)^p|y|^{-Np}\,dy'\,dy_N \\
&\le Cx_N|x|^{-N}\int_0^\infty\int_{\frac{3|x'|}{2}+1+x_N}^\infty (r+y_N)^{-Np+p+N-1}\,dr\,dy_N \\
&\le Cx_N|x|^{-N}\int_0^\infty (|x|+y_N)^{-Np+p+N}\,dy_N \le Cx_N|x|^{-N}.
\end{aligned}$$

This implies (4.24) with $k = 7$. Thus (4.24) holds for $k = 1, \dots, 7$, and the proof of Lemma 4.2 is complete. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We proceed similarly as in the proof of Theorem 1.4 in [10]. Define $\{u_n\}$ inductively by

$$\begin{aligned} &u_1(x) := 0, \\ &u_{n+1}(x) := [S(x_N)\varphi](x') + \int_{\mathbb{R}^N_+} G(x,y)u_n(y)^p\,dy, \quad n = 1, 2, \dots, \end{aligned}$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$. Then we can prove by induction that

$$0 \leq u_{n-1}(x) \leq u_n(x)$$

for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$. This means that the limit function

$$u(x) := \lim_{n\to\infty} u_n(x) \in [0, \infty] \tag{4.50}$$

can be defined for almost all $x' \in \mathbb{R}^{N-1}$ and all $x_N \geq 0$.

Let c and k be sufficiently small positive constants such that $c < k$. Assume that

$$\varphi(x') \geq c\psi(x'), \qquad x' \in \mathbb{R}^{N-1},$$

where ψ is the function given in (1.6). It follows from Lemma 3.1 that

$$u(x) \geq [S(x_N)\varphi](x') \geq c[S(x_N)\psi](x') \geq C_1 c \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}}, \\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases} \tag{4.51}$$

where C_1 is a positive constant independent of c. On the other hand, if φ satisfies

$$\varphi(x') \leq k\psi(x'), \qquad x' \in \mathbb{R}^{N-1},$$

then, by Lemma 3.2 we see that

$$u_2(x) = [S(x_N)\varphi](x) \leq k[S(x_N)\psi](x') \leq C_2 k \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}}, \\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases} \tag{4.52}$$

where C_2 is a positive constant independent of k. If

$$u_n(x) \leq 2C_2 k \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}}, \\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases}$$

for some $n \in \{2, 3, \dots\}$, then, by Lemmas 4.1 and 4.2 we see that

$$\begin{aligned} u_{n+1}(x) &= [S(x_N)\varphi](x') + [(-\Delta_D)^{-1}u_n^p](x) \\ &\leq [C_2 k + 2C_3(2C_2 k)^p] \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}}, \\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}, \end{cases} \end{aligned}$$

where C_3 is a positive constant independent of k. Therefore, taking a sufficiently small constant k if necessary, we have

$$u_{n+1}(x) \leq 2C_2 k \begin{cases} |x|^{-m}, & x \in D_{\mathrm{in}}, \\ (1 + x_N)|x|^{-N}, & x \in D_{\mathrm{out}}. \end{cases}$$

This together with (4.50) and (4.52) implies that

$$u(x) \leq 2C_2 k \begin{cases} |x|^{-m}, & x \in D_{\rm in}, \\ (1+x_N)|x|^{-N}, & x \in D_{\rm out}. \end{cases} \tag{4.53}$$

Therefore, if φ satisfies (1.7) with sufficiently small constants c and k, then (4.51) and (4.53) imply that u satisfies (1.8) and (1.9). Then, similarly to [11], we see that u is a minimal solution of (1.1). Thus Theorem 1.1 follows. □

References

[1] I. Bachar and H. Mâagli, *Estimates on the Green's function and existence of positive solutions of nonlinear singular elliptic equations in the half space,* Positivity, **9** (2005), 153–192.

[2] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems,* Topol. Methods Nonlinear Anal., **4** (1994), 59–78.

[3] M.-F. Bidaut-Véron, A. Ponce and L. Véron, *Isolated boundary singularities of semilinear elliptic equations,* Calc. Var. Partial Differential Equations, **40** (2011), 183–221.

[4] M.-F. Bidaut-Véron and L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case,* Rev. Mat. Iberoam., **16** (2000), 477–513.

[5] C.-C. Chen and C.-S. Lin, *Existence of positive weak solutions with a prescribed singular set of semilinear elliptic equation,* J. Geom. Anal., **9** (1999), 221–246.

[6] M. del Pino, M. Musso and F. Pacard, *Boundary singularities for weak solutions of semilinear elliptic problems,* J. Funct. Anal., **253** (2007), 241–272.

[7] A. Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of* $\mathbb{R}^N$, J. Math. Pures Appl., **87** (2007), 537–561.

[8] M. Fila, K. Ishige and T. Kawakami, *Large-time behavior of solutions of a semilinear elliptic equation with a dynamical boundary condition,* Adv. Differential Equations, **18** (2013), 69–100.

[9] M. Fila, K. Ishige and T. Kawakami, *Large-time behavior of small solutions of a two-dimensional semilinear elliptic equation with a dynamical boundary condition,* Asymptotic Analysis, **85** (2013), 107–123.

[10] M. Fila, K. Ishige and T. Kawakami, *Existence of positive solutions of a semilinear elliptic equation with a dynamical boundary condition,* Calc. Var. Partial Differential Equations, **54** (2015), 2059–2078.

[11] M. Fila, K. Ishige and T. Kawakami, *Minimal solutions of a semilinear elliptic equation with a dynamical boundary condition,* J. Math. Pures Appl., to appear, doi:10.1016/j.matpur.2015.11.014.

[12] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations,* Comm. Partial Differential Equations, **6** (1981), 883–901.

[13] A. Gmira and L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations,* Duke Math. J., **64** (1991), 271–324.

[14] K. Ishige, K. Kawakami and K. Kobayashi, *Global solutions for a nonlinear integral equation with a generalized heat kernel,* Discrete Contin. Dyn. Syst. Ser. S., **7** (2014), 767–783.

[15] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case,* Arch. Ration. Mech. Anal., **144** (1998), 201–231.

[16] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case,* J. Math. Pures Appl., **77** (1998), 481–524.

[17] M. Marcus and L. Véron, *Removable singularities and boundary traces,* J. Math. Pures Appl., **80** (2001), 879–900.

[18] Ph. Souplet, *Optimal regularity conditions for elliptic problems via L^p_δ-spaces,* Duke Math. J., **127** (2005), 175–192.

Department of Applied Mathematics and Statistics, Comenius University, 84248 Bratislava, Slovakia
E-mail address: `fila@fmph.uniba.sk`

Mathematical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan
E-mail address: `ishige@math.tohoku.ac.jp`

Department of Mathematical Sciences, Osaka Prefecture University, Sakai 599-8531, Japan
E-mail address: `kawakami@ms.osakafu-u.ac.jp`

JEPE Vol 1, 2015, p. 363-385

ON THE FINITE TIME BLOW-UP OF BIHARMONIC MAP FLOW IN DIMENSION FOUR

LEI LIU AND HAO YIN

ABSTRACT. In this paper, we show that for certain initial values, the (extrinsic) biharmonic map flow in dimension four must blow up in finite time.

1. INTRODUCTION

Let (M,g) be a compact Riemannian manifold without boundary of dimension four and (N,h) be another compact Riemannian manifold without boundary, which is isometrically embedded in $\mathbb{R}^K$. The critical points of the following functional

$$E(u) = \int_M |\triangle u|^2 \, dv$$

are called (extrinsic) biharmonic maps. We also define

$$\mathcal{E}(u) = \int_M \left|\nabla^2 u\right|^2 + |\nabla u|^4 \, dv$$

and notice that since the target manifold is compact, we can bound $\mathcal{E}(u)$ by $E(u)$.

The associated heat flow of $E(u)$ was first studied by Lamm [8]. In [8], the author proved that in dimension four, the following evolution equation

$$\partial_t u = -\triangle^2 u + \triangle(B(u)(\nabla u, \nabla u)) + 2\nabla\langle \triangle u \nabla P(u)\rangle - \langle \triangle P(u), \triangle u\rangle \tag{1.1}$$

has a unique short time smooth solution for all smooth initial value. Here B is the second fundamental form of $N \subset \mathbb{R}^K$ and $P(u)$ is the projection from $\mathbb{R}^K$ to the tangent space $T_u N$. Moreover, the solution is global if the $W^{2,2}$ norm of the initial value is small. Following the famous work of Struwe on harmonic map flow [13], Gastel [7] and Wang [17] showed the existence of a global weak solution with at most finitely many singular times.

It is a natural question whether the flow develops finite time singularity. The problem is particularly interesting given that all weak biharmonic maps with bounded $W^{2,2}$ norm in dimension four are known to be smooth (see [15]). The corresponding problem for harmonic map flow was answered by Chang, Ding and Ye [3]. After that, more finite-time singularity examples were found by Topping [14], Li and Wang [9] and very recently by Chen and Li [4]. The last construction shows that the blow-up could be forced by topological reason and its proof relies on the no neck theorem for approximate harmonic maps of Qing and Tian

2010 *Mathematics Subject Classification.* Primary: 58E20; Secondary: 35J50 53C43.

Key words and phrases. Biharmonic map flow, finite blow-up, neck analysis.
Received 01/06/2015, accepted 13/10/2015.
The research of the second author is supported by NSFC 11101272 and NSFC 11471300.

[11]. In fact, it was pointed out by Qing and Tian that the no neck theorem could be used in showing finite time blow-up.

Recently, the authors proved the no neck theorem for the blow-up of a sequence of (extrinsic) biharmonic maps with bounded energy. The proof follows from an idea of [11], which was motivated by the three circle lemma of Simon [12]. In light of [4], it is very natural to extend the argument to the case of biharmonic map flow and this is the purpose of this paper. More precisely, we show

Theorem 1. *Suppose that N' is any closed manifold of dimension $m > 4$ with nontrivial $\pi_4(N')$ and let $N = N'\#T^m$ be the connected sum of N' with the torus of the same dimension. For any Riemannian metric g on N, we can find (infinitely many) initial map $u_0 : S^4 \to N$ such that the biharmonic map flow (1.1) starting from u_0 develops finite time singularity.*

As remarked earlier, the proof relies on the idea of [4]. However, we give a slightly different presentation. Our assumption on the topology of N enables us to be more specific in the construction. Moreover, we define and use the concept of the width of a biharmonic map u from S^4 to N. Roughly speaking, the idea of the proof is the following. By a compactness argument, we show that the width of biharmonic maps from S^4 to N is bounded by a constant depending on the energy of the map (and the geometry of N of course). However, we can construct initial map u_0 with bounded energy but in a homotopy class in which every smooth representation must have very large width. If no finite-time singularity occurs, we may choose a sequence $t_i \to \infty$ such that the bi-tension field of $u(t_i)$ goes to zero in L^2 norm. Hence, $u(t_i)$ is a sequence of approximate biharmonic maps. $u(t_i)$ either converges to a smooth biharmonic map in the same homotopy class, which is not possible because the energy of the limit is smaller than that of u_0, or blows up. In the latter case, the total number of bubbles and energy of each bubble, as well as the weak limit is bounded and the no neck theorem (Theorem 2) implies a contradiction as well.

The rest of the paper is organized as follows. In Section 2, we generalize the no neck result in [10] to the case of approximate biharmonic maps. The generalization is in two directions. The first is to involve a non-zero bi-tension field and the second is to show the neck analysis works on round sphere instead of flat domains in $\mathbb{R}^4$.

Remark 1. For many PDE theorems, especially about regularity of geometric PDE, the curvature of the domain is not essential. Hence, it suffices to prove the theorem in the case of domains of Euclidean space. In this paper, we think it may not be very obvious that the neck analysis of biharmonic maps works on curved space. Hence, we present a detailed proof in the case of round metric on S^4, which is needed by the proof of Theorem 1.

In spite of the complexity caused by the round metric, we still believe that the neck analysis works in general. However, that would require greater efforts. We also note that this is not an issue for the neck analysis of harmonic maps, because of the conformal invariance.

In Section 3, the width of a map from S^4 to N is defined and the width of biharmonic maps from both S^4 and $\mathbb{R}^4$ are bounded by the energy. Finally, Theorem 1 is proved in Section 4.

Remark 2. Recently, we notice that Breiner and Lamm [2] proved a no neck theorem for a sequence of biharmonic maps with bi-tension fields in $L\log L$ when the target manifold is a sphere. In this paper, by approximate biharmonic maps, we mean bi-tension field is bounded in L^2.

We also want to mention a paper of Cooper [5]. The author constructed blow-up solution from B^4 to S^4 by using symmetry.

2. No neck for approximate biharmonic maps

In this section, we show that the main result of [10] can be generalized to a sequence of approximate biharmonic maps u_i defined on S^4.

We use a subscript g to denote operators defined on S^4 with round metric, such as $\triangle_g$ and ∇_g. $\triangle$ and ∇ are reserved for the Laplace and gradient with respect to the flat metric given by normal coordinates around some point in S^4. We always take the normal coordinates x so that the scaling $u(\lambda x)$ is well defined for small λ. Moreover, due to the Gauss Lemma, the geodesic ball B_r is the same as the ball of radius r with respect to the flat metric given by the normal coordinates. Finally, there is no need to distinguish the L^p norms defined with round metric g and the flat metric in normal coordinates for our purpose.

We will prove

Theorem 2. *Let u_i be a sequence of approximate biharmonic maps from B^4 to N satisfying*

$$\triangle_g^2 u = \triangle_g(B(u)(\nabla_g u, \nabla_g u)) + 2\nabla_g \cdot \langle \triangle_g u, \nabla(P(u))\rangle - \langle \triangle_g(P(u)), \triangle_g u\rangle + \tau(u). \tag{2.1}$$

with

$$\int_{B_1} \left|\nabla_g^2 u_i\right|^2 + \left|\nabla_g u_i\right|^4 dv_g < \Lambda \quad \text{and} \quad \|\tau(u_i)\|_{L^p(B_1)} < \Lambda \tag{2.2}$$

for some $\Lambda > 0$ and $p \geq \frac{4}{3}$. Assume that there is a positive sequence $\lambda_i \to 0$ such that

$$u_i(\lambda_i x) \to \omega$$

in $W^{4,p} \subset W^{2,2} \cap C^0$ on any compact set $K \subset \mathbb{R}^4$, that u_i converges weakly in $W^{2,2}$ to u_∞ and that ω is the only bubble. Then,

$$\lim_{\delta\to 0} \lim_{R\to\infty} \lim_{i\to\infty} osc_{B_\delta(0)\setminus B_{\lambda_i R}(0)} u_i = 0. \tag{2.3}$$

Remark 3. In Theorem 2, we assume that there is only one bubble. The same result holds in the case of multiple bubbles. The proof is routine argument by now and hence is omitted.

The proof is similar to the proof of Theorem 1.1 in [10], which we outline below. We first recall some definitions and results, which are modified only slightly.

2.1. **Minor modifications.**

The following theorem is a modified version of $\varepsilon-$regularity, proved in the Appendix of [10].

Theorem 3 (ε_0-regularity)**.** *Let $u \in W^{4,p}(B_1) (p > 1)$ be an approximate biharmonic map. There exists $\varepsilon_0 > 0$ such that if $\int_{B_1} |\nabla^2 u|^2 + |\nabla u|^4 dx \leq \epsilon_0$ then*

$$\|u - \overline{u}\|_{W^{4,p}(B_{1/2})} \leq C(\|\nabla^2 u\|_{L^2(B_1)} + \|\nabla u\|_{L^4(B_1)} + \|\tau(u)\|_{L^p(B_1)}),$$

where $\overline{u}$ is the mean value of u over B_1.

Remark 4. We may very well use ∇_g in the above lemma. It is the type of result that Riemannian metric does not make any difference.

Next, we modify the definition of $\eta-$approximate biharmonic map as follows.

Definition 1. Let u be a smooth function defined on $B_{r_2}\backslash B_{r_1}$, u is called an $\eta-$approximate biharmonic function if it satisfies

$$\begin{aligned}\triangle_g^2 u(r,\theta) &= a_1\nabla_g\triangle_g u + a_2\nabla_g^2 u + a_3\nabla_g u + a_4 u \\ &\quad + \frac{1}{|\partial B_r|}\int_{\partial B_r} b_1\nabla_g\triangle_g u + b_2\nabla_g^2 u + b_3\nabla_g u + b_4 u d\sigma + h(x).\end{aligned}\tag{2.4}$$

where a_i, b_i and h are smooth functions satisfying, for any $\rho\in[r_1, r_2/2]$,

(a) $\|g_{ij}(\rho x)-\delta_{ij}\|_{C^4(B_2\backslash B_1)}<\eta$. Namely, the metric after scaling to $B_2\setminus B_1$ is close to the flat metric in C^4 norm.

(b)

$$\||x|^{4(1-1/p)}h\|_{L^p(B_{r_2}\backslash B_{r_1})}\le\eta \tag{2.5}$$

and

(c)

$$\sum_{i=1}^4 \|a_i\|_{\tilde W^{4-i,p}(B_{2\rho}\backslash B_\rho)} + \|b_i\|_{\tilde W^{4-i,p}(B_{2\rho}\backslash B_\rho)}\le\eta.$$

Here $\tilde W^{4-i,p}$ norm is defined as

$$\|w\|_{\tilde W^{4-i,p}(B_{2\rho}\backslash B_\rho)} = \left\|\rho^i w(\rho x)\right\|_{W^{4-i,p}(B_2\backslash B_1)}.$$

Remark 5. The condition (c) implies that if we compute the equation satisfied by $\tilde u(x) = u(\rho x)$ (defined on $B_2\setminus B_1$), then the coefficients appearing in the place of a_i (or b_i) become $\rho^i a_i(\rho x)$ (or $\rho^i b_i(\rho x)$), which is assumed to be small in $W^{4-i,p}$ norm.

Moreover, one can check that if u is an $\eta-$approximate biharmonic function on $B_{r_2}\setminus B_{r_1}$, then $w(x)=u(\frac{x}{\lambda})$ is also an $\eta-$approximate biharmonic function on $B_{\lambda r_2}\setminus B_{\lambda r_1}$.

The following is a version of interior L^p estimate for approximate biharmonic function. It is used in the proof of three circle lemma.

Lemma 1. *There is some $\eta_0>0$ such that for any $\eta<\eta_0$ if $u: B_4\setminus B_1\to\mathbb{R}$ is a $\eta-$approximate biharmonic function with*

$$\sum_{i=1}^4 \|a_i\|_{W^{4-i,p}(B_4\backslash B_1)} + \|b_i\|_{W^{4-i,p}(B_4\backslash B_1)}\le\eta \quad \textit{and} \quad \|h\|_{L^p(B_4\backslash B_1)}\le C.$$

Then, for any $4>p>1$, we have

$$\|u\|_{W^{4,p}(B_3\backslash B_2)}\le C(\|u\|_{L^p(B_4\backslash B_1)}+\|h\|_{L^p(B_4\backslash B_1)}).$$

Proof. Without loss of generality, we assume the metric g is the standard Euclidean metric. The main idea is similar to the lemma 3.3 in [10], but the assumptions on a_i and b_i are different from [10]. Next, we sketch the proof here.

For $0<\sigma<1$, set $A_\sigma = B_{3+\sigma}\setminus B_{2-\sigma}$ and $A'_\sigma = B_{3+\frac{1+\sigma}{2}}\setminus B_{2-\frac{1+\sigma}{2}}$. Let φ be a cut-off function supported in A'_σ satisfying: (1) $\varphi\equiv 1$ in A_σ; (2) $\left|\nabla^j\varphi\right|\le\frac{c}{(1-\sigma)^j}$ for $j=1,2,3,4$ and some universal constant c; (3) φ is a function of $|x|$.

Computing directly, we have

$$\begin{aligned}
\triangle^2(\varphi u) &= \triangle(\varphi\triangle u + 2\nabla\varphi\nabla u + u\triangle\varphi)\\
&= \varphi\triangle^2 u + 4\nabla\triangle u\nabla\varphi + 4\nabla^2 u\nabla^2\varphi + 2\triangle u\triangle\varphi + 4\nabla\triangle\varphi\nabla u + \triangle^2\varphi u\\
&= \varphi a_1\nabla\triangle u + \varphi a_2\nabla^2 u + \varphi a_3\nabla u + \varphi a_4 u + \varphi h\\
&\quad +\varphi\frac{1}{|\partial B_r|}\int_{\partial B_r} b_1\nabla\triangle u + b_2\nabla^2 u + b_3\nabla u + b_4 u d\sigma\\
&\quad +4\nabla\triangle u\nabla\varphi + 4\nabla^2 u\nabla^2\varphi + 2\triangle u\triangle\varphi + 4\nabla\triangle\varphi\nabla u + \triangle^2\varphi u.
\end{aligned}$$

Next, we estimate the $L^p(p>1)$ norm of the right hand side of the above equation. By our choice of φ and the assumption of a_1, we have

$$\|\nabla\triangle u\nabla\varphi\|_{L^p(A'_\sigma)} \leq \frac{C}{1-\sigma}\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}$$

and

$$\begin{aligned}
&\|\varphi a_1\nabla\triangle u\|_{L^p(A'_\sigma)}\\
\leq\ & \|a_1\nabla\triangle(\varphi u)\|_{L^p(A'_\sigma)} + \left\|a_1\nabla^2 u\nabla\varphi\right\|_{L^p(A'_\sigma)} + \left\|a_1\nabla u\nabla^2\varphi\right\|_{L^p(A'_\sigma)} + \left\|a_1 u\nabla^3\varphi\right\|_{L^p(A'_\sigma)}\\
\leq\ & C\|a_1\|_{W^{3,p}}\left\|\nabla^3(\varphi u)\right\|_{W^{1,p}(A'_\sigma)} + \frac{C}{1-\sigma}\left\|a_1\nabla^2 u\right\|_{L^p(A'_\sigma)} + \frac{C}{(1-\sigma)^2}\|a_1\nabla u\|_{L^p(A'_\sigma)}\\
& +\frac{C}{(1-\sigma)^3}\|a_1 u\|_{L^p(A'_\sigma)}\\
\leq\ & C\eta\|\varphi u\|_{W^{4,p}(A'_\sigma)} + C\left(\frac{\left\|\nabla^2 u\right\|_{W^{1,p}(A'_\sigma)}}{1-\sigma} + \frac{\|\nabla u\|_{W^{1,p}(A'_\sigma)}}{(1-\sigma)^2} + \frac{\|u\|_{W^{1,p}(A'_\sigma)}}{(1-\sigma)^3}\right)\\
\leq\ & C\eta\|\varphi u\|_{W^{4,p}(A'_\sigma)} + C\left(\frac{\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}}{1-\sigma} + \frac{\left\|\nabla^2 u\right\|_{L^p(A'_\sigma)}}{(1-\sigma)^2} + \frac{\|\nabla u\|_{L^p(A'_\sigma)}}{(1-\sigma)^3} + \frac{\|u\|_{L^p(A'_\sigma)}}{(1-\sigma)^4}\right).
\end{aligned}$$

Here in the third and fourth line above, we use the Sobolev multiplication theorem which says that $\|f_1 f_2\|_{L^p} \leq C\|f_1\|_{W^{4-i,p}}\|f_2\|_{W^{i,p}}$.

Moreover, Jensen's inequality implies that

$$\begin{aligned}
&\int_{A'_\sigma}\frac{\varphi^p}{|\partial B_r|^p}\left(\int_{\partial B_r} b_1\nabla\triangle u\right)^p dx\\
\leq\ & \int_{A'_\sigma}\varphi^p\frac{1}{|\partial B_r|}\left(\int_{\partial B_r}|b_1\nabla\triangle u|^p\right)dx\\
\leq\ & C\int_{A'_\sigma}\varphi^p\left|b_1\nabla^3 u\right|^p dx.
\end{aligned}$$

Now, the same estimate used for $\|\varphi a_1\nabla\triangle u\|_{L^p(A'_\sigma)}$ can be used again to get the same upper bound.

Similar argument applies to the remaining terms and gives an estimate of L^p norm of $\triangle^2(\varphi u)$, if we choose η sufficiently small, by which the L^p estimate of bi-Laplace operator implies

$$\|\varphi u\|_{W^{4,p}(A'_\sigma)} \leq C\left(\frac{\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}}{1-\sigma} + \frac{\left\|\nabla^2 u\right\|_{L^p(A'_\sigma)}}{(1-\sigma)^2} + \frac{\|\nabla u\|_{L^p(A'_\sigma)}}{(1-\sigma)^3} + \frac{\|u\|_{L^p(A'_\sigma)}}{(1-\sigma)^4} + \|h\|_{L^p}\right).$$

In particular, we have

$$\begin{aligned}(1-\sigma)^4\left\|\nabla^4 u\right\|_{L^p(A_\sigma)} &\leq C\Big((1-\sigma)^3\left\|\nabla^3 u\right\|_{L^p(A'_\sigma)}+(1-\sigma)^2\left\|\nabla^2 u\right\|_{L^p(A'_\sigma)}\\ &\quad +(1-\sigma)\|\nabla u\|_{L^p(A'_\sigma)}+\|u\|_{L^p(A'_\sigma)}+\|h\|_{L^p}\Big).\end{aligned}$$

By setting

$$\Psi_j=\sup_{0\leq\sigma\leq 1}(1-\sigma)^j\left\|\nabla^j u\right\|_{L^p(A_\sigma)}$$

and noting that

$$A'_\sigma=A_{\frac{1+\sigma}{2}} \quad \text{and} \quad 1-\sigma=2(1-\frac{1+\sigma}{2}),$$

we obtain

$$\Psi_4\leq C(\Psi_3+\Psi_2+\Psi_1+\Psi_0+\|h\|_{L^p}). \tag{2.6}$$

We claim that for $j=1,2,3$, the following interpolation inequality holds for any $\epsilon>0$,

$$\Psi_j\leq\epsilon^{4-j}\Psi_4+\frac{C}{\epsilon^j}\Psi_0.$$

In fact, by the definition of Ψ_j, for any $\gamma>0$, there is $\sigma_\gamma\in[0,1]$ such that

$$\begin{aligned}\Psi_j &\leq (1-\sigma_\gamma)^j\left\|\nabla^j u\right\|_{L^p(A_{\sigma_\gamma})}+\gamma\\ &\leq \epsilon^{4-j}(1-\sigma_\gamma)^4\left\|\nabla^4 u\right\|_{L^p(A_{\sigma_\gamma})}+\frac{C}{\epsilon^j}\|u\|_{L^p(A_{\sigma_\gamma})}+\gamma\\ &\leq \epsilon^{4-j}\Psi_4+\frac{C}{\epsilon^j}\Psi_0+\gamma.\end{aligned}$$

Here we used the interpolation inequality

$$\left\|\nabla^j u\right\|_{L^p(A_{\sigma_\gamma})}\leq\eta^{4-j}\left\|\nabla^4 u\right\|_{L^p(A_{\sigma_\gamma})}+\frac{C_3}{\eta^j}\|u\|_{L^p(A_{\sigma_\gamma})} \tag{2.7}$$

with $\eta=\epsilon(1-\sigma_\gamma)$. We remark that the constant in the above interpolation inequality are independent of $\sigma\in[0,1]$ (see the proof of Lemma 5.6 in [1]).

By sending γ to 0 and choosing small ϵ, we obtain from (2.6)

$$\Psi_4\leq C\left(\Psi_0+\|h\|_{L^p}\right),$$

from which our lemma follows. □

For the universal constant $L>0$ given in Section 3 of [10], set

$$A_i=B_{e^{-(i-1)L}}\setminus B_{e^{-iL}}$$

and

$$F_i(u)=\int_{A_i}\frac{1}{|x|^4}u^2dx.$$

Remark 6. Here is a technical issue. We use dx instead of dv_g in the definition of $F_i(u)$. The advantage is that $F_i(u)$ is invariant under the scaling $x\to\lambda x$. Since g is close to Euclidean metric, this difference does not matter when we use $F_i(u)$ as a control of L^2 norm.

Theorem 4. *There is some constant $\eta_0 > 0$ such that the following is true. Assume that $u : A_{i-1} \cup A_i \cup A_{i+1} \to \mathbb{R}^K$ is an η_0-approximate biharmonic function in the sense of (2.4). Suppose*

$$\max_{i-1,i,i+1} \||x|^{4(1-1/p)} h\|^2_{L^p(A_i)} \le \eta_0 F_i(u) \tag{2.8}$$

and

$$\int_{\partial B_r} u d\theta = 0 \tag{2.9}$$

for $r \in [e^{-l_1 L}, e^{-(l_2-1)L}]$. Then
(a) if $F_{i+1}(u) \le e^{-L} F_i(u)$, then $F_i(u) \le e^{-L} F_{i-1}(u)$;
(b) if $F_{i-1}(u) \le e^{-L} F_i(u)$, then $F_i(u) \le e^{-L} F_{i+1}(u)$;
(c) either $F_i(u) \le e^{-L} F_{i-1}(u)$, or $F_i(u) \le e^{-L} F_{i+1}(u)$.

Proof. (The proof is almost the same as Theorem 3.4 in [10]. For reader's convenience, we repeat it below.)

The exact value of i does not matter, because F_i is invariant under scaling. Hence, we consider only the case of $i = 2$. Assume the theorem is not true. We have a sequence of $\eta_k \to 0$ and a sequence of u_k defined on $A_1 \cup A_2 \cup A_3$ (and a sequence of g_k defined on $A_1 \cup A_2 \cup A_3$ as required in (a) of Definition 1) satisfying

$$\begin{aligned} \triangle^2_{g_k} u_k(r,\theta) &= a_{k1} \nabla_{g_k} \triangle_{g_k} u_k + a_{k2} \nabla^2_{g_k} u_k + a_{k3} \nabla_{g_k} u_k + a_{k4} u_k \\ &+ \frac{1}{|\partial B_r|} \int_{\partial B_r} b_{k1} \nabla_{g_k} \triangle_{g_k} u_k + b_{k2} \nabla^2_{g_k} u_k + b_{k3} \nabla_{g_k} u_k + b_{k4} u_k + h_k(x) \end{aligned} \tag{2.10}$$

with

$$\max_{1,2,3} \||x|^{4(1-1/p)} h_k\|^2_{L^p(A_i)} \le \eta_k F_2(u_k) \tag{2.11}$$

and

$$\sum_{i=1}^{4} \|a_{ki}\|_{\tilde{W}^{4-i,p}(B_{2\rho} \setminus B_\rho)} + \|b_{ki}\|_{\tilde{W}^{4-i,p}(B_{2\rho} \setminus B_\rho)} \le \eta_k, \tag{2.12}$$

for any $B_{2\rho} \setminus B_\rho \subset A_1 \cup A_2 \cup A_3$.

By taking subsequence, we assume that one of (a), (b) and (c) is not true for u_k. If (a) is not true, then we have

$$F_2(u_k) \ge e^L F_3(u_k) \quad \text{and} \quad F_2(u_k) > e^{-L} F_1(u_k).$$

If (b) is not true, then

$$F_2(u_k) \ge e^L F_1(u_k) \quad \text{and} \quad F_2(u_k) > e^{-L} F_3(u_k).$$

If (c) is not true, then

$$F_2(u_k) > e^{-L} \max\{F_1(u_k), F_3(u_k)\}.$$

In any case, we control $F_1(u_k)$ and $F_3(u_k)$ by $F_2(u_k)$. Multiplying by a constant to u_k if necessary, we assume that $F_2(u_k) = 1$ for all k. The above discussion shows that

$$\|u_k\|_{L^2(A_1 \cup A_2 \cup A_3)} \le C.$$

Lemma 1 shows that (by passing to a subsequence) we have

$$\begin{aligned} u_k &\rightharpoonup u \qquad \textit{weakly} \quad \textit{in} \quad L^2(A_1\cup A_2\cup A_3),\\ u_k &\to u \qquad \textit{strongly} \quad \textit{in} \quad L^2(A_2). \end{aligned}$$

By (2.10), (2.11) and (2.12), we know that u is a nonzero biharmonic function with respect to the flat metric defined on $A_1\cup A_2\cup A_3$ satisfying (2.9), because g_k converges strongly in C^3 norm to the flat metric. The three circle lemma for biharmonic function (Theorem 3.1 in [10]) implies that

$$2F_2(u) < e^{-L}(F_1(u)+F_3(u)). \tag{2.13}$$

If (c) does not hold for u_k, we have

$$2F_2(u_k) \ge e^{-L}(F_1(u_k)+F_3(u_k)).$$

By the strong convergence of u_k in $L^2(A_2)$ and weak convergence in $L^2(A_1\cup A_2\cup A_3)$, we have

$$2F_2(u) \ge e^{-L}(F_1(u)+F_3(u)),$$

which is a contradiction to (2.13). Similar argument works for other cases. □

2.2. **Estimate of the tangential energy.** Let u_i be the sequence in Theorem 2. Assume that

$$\Sigma = B_\delta \setminus B_{\lambda_i R} = \bigcup_{l=l_0}^{l_i} A_l$$

and for any $\varepsilon>0$, by choosing δ small and R large, we may also assume (by an induction argument of Ding and Tian [6])

$$\int_{A_l} \left|\nabla^2 u_i\right|^2 + |\nabla u_i|^4\, dx < \varepsilon^4 < \varepsilon_0 \tag{2.14}$$

for $l=l_0,\cdots,l_i$. Set $\widetilde{u_i}(x)=u_i(e^{-lL}x)$, by ε_0-regularity Theorem 3, we have

$$\|\widetilde{u_i}-\overline{\widetilde{u_i}}\|_{W^{4,p}(A_0)} \le C(\|\nabla^2\widetilde{u_i}\|_{L^2(A_{-1}\cup A_0\cup A_1)}+\|\nabla\widetilde{u_i}\|_{L^4(A_{-1}\cup A_0\cup A_1)}+\|\tau(\widetilde{u_i})\|_{L^p(A_{-1}\cup A_0\cup A_1)}),$$

where $\overline{\widetilde{u_i}}$ is the mean value of $\widetilde{u_i}$ over A_0.

Scaling back, if δ is sufficiently small, we will get

$$\begin{aligned} &\sum_{k=0}^{4} \||x|^{k-4/p}\nabla^k(u_i-\overline{\widetilde{u_i}})\|_{L^p(A_l)}\\ \le\ & C(\|\nabla^2 u_i\|_{L^2(A_{l-1}\cup A_l\cup A_{l+1})}+\|\nabla u_i\|_{L^4(A_{l-1}\cup A_l\cup A_{l+1})}+e^{-lL4(1-1/p)}\|\tau(u_i)\|_{L^p(A_{l-1}\cup A_l\cup A_{l+1})})\\ \le\ & C\varepsilon. \end{aligned}$$

Let $r=e^t$, then as a function of (t,θ), we have

$$\|u_i-\overline{\widetilde{u_i}}\|_{W^{4,p}(-lL,-(l-1)L)\times S^3} \le C\varepsilon, \tag{2.15}$$

for any $l_0\le l\le l_i$.

The theorem is equivalent to the statement that for any $\varepsilon>0$, we can find δ small and R large such that

$$\operatorname{osc}_{B_\delta\setminus B_{\lambda_i R}} u_i < C\varepsilon$$

for i sufficiently large.

Set

$$u_i^*(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} u_i(r,\theta)d\sigma.$$

The Poincaré inequality and (2.14) imply

$$\int_{A_l} \frac{1}{|x|^4}\left|u_i - u_i^*\right|^2 dx \le C\varepsilon^2.$$

Lemma 2. *There exists some $\varepsilon_1 > 0$ such that if $\varepsilon < \varepsilon_1$ in (2.14) and $\delta < \varepsilon_1$, $w_i = u_i - u_i^*$ is an η_0-approximate biharmonic function defined on $B_\delta \setminus B_{\lambda_i R}$ in the sense of (2.4), where η_0 is the constant in Theorem 4.*

Remark 7. Although the proof is parallel to Lemma 4.1 in [10]. We reproduce it because (1) we now use the sphere metric instead of the flat one; (2) the definition of $\eta-$approximate biharmonic function is different.

Proof. For simplicity, we omit the subscript i. Recall that u satisfies

$$\begin{aligned}\triangle_g^2 u &= \alpha_1(u)\nabla_g\triangle_g u\#\nabla_g u + \alpha_2(u)\nabla_g^2 u\#\nabla_g^2 u\\ &+\alpha_3(u)\nabla_g^2 u\#\nabla_g u\#\nabla_g u + \alpha_4(u)\nabla_g u\#\nabla_g u\#\nabla_g u\#\nabla_g u + \tau(u).\end{aligned}\tag{2.16}$$

Here $\alpha_i(u)$ is a smooth function of u and $\#$ is the contraction of tensors with respect to g, for which we have for example,

$$\left|\nabla_g\triangle_g u\#\nabla_g u\right| \le C\left|\nabla_g\triangle_g u\right|\left|\nabla_g u\right|.$$

Since $\triangle_g = \frac{\partial^2}{\partial r^2} + \frac{3\cos r}{\sin r}\frac{\partial}{\partial r} + \frac{1}{\sin^2 r}\triangle_{S^3}$ and $\int_{S^3}\triangle_{S^3} f d\theta = 0$ for any f, we have

$$\begin{aligned}\triangle_g^2 u^*(r) &= \frac{1}{|\partial B_r|}\int_{\partial B_r}\triangle_g^2 u d\sigma\\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r}\alpha_1(u)\nabla_g\triangle_g u\#\nabla_g u + \alpha_2(u)\nabla_g^2 u\#\nabla_g^2 u\\ &\quad +\alpha_3(u)\nabla_g^2 u\#\nabla_g u\#\nabla_g u + \alpha_4(u)\nabla_g u\#\nabla_g u\#\nabla_g u\#\nabla_g u d\sigma\\ &\quad +\frac{1}{|\partial B_r|}\int_{\partial B_r}\tau(u)d\sigma\\ &= I + II + III + IV + \frac{1}{|\partial B_r|}\int_{\partial B_r}\tau(u)d\sigma.\end{aligned}$$

Remark 8. Here we make essential use of the symmetry of spherical metric to simplify the computation in the first line above. This is partially the reason that we work on round S^4.

Computing directly, we get

$$\begin{aligned}I &= \frac{1}{|\partial B_r|}\int_{\partial B_r}\alpha_1(u)\nabla_g\triangle_g u\#\nabla_g u - \alpha_1(u^*)\nabla_g\triangle_g u\#\nabla_g u\\ &\quad +\alpha_1(u^*)\nabla_g\triangle_g u\#\nabla_g u - \alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u\\ &\quad +\alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u - \alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u^* d\sigma\\ &\quad +\alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u^*\\ &= \frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_4[u](u-u^*) + \beta_1[u]\nabla_g\triangle_g(u-u^*)\\ &\quad +\beta_3[u]\nabla_g(u-u^*)d\sigma + \alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u^*.\end{aligned}$$

Here $\beta_i[u]$ is some expression depending on u, u^* and their derivatives. Precisely, we have

$$\begin{aligned}\beta_4[u] &= \alpha_1'(*)\nabla_g\triangle_g u\#\nabla_g u\\ \beta_1[u] &= \alpha_1(u^*)\nabla_g u\\ \beta_3[u] &= \alpha_1(u^*)\nabla_g\triangle_g u^*.\end{aligned}$$

In what follows, those β_i's may differ from line to line in the following. However, thanks to Theorem 3, we have

$$\|\beta_i\|_{\tilde{W}^{4-i,p}(B_{2\rho}\setminus B_\rho)} \le \eta_0 \quad \text{for} \quad \rho\in[\lambda_i R,\delta/2],$$

if ε in (2.14) is smaller than some ε_1. We shall require the above holds for all β_i and β_i' below by asking ε_1 to be smaller and smaller.

The same computation gives

$$II = \frac{1}{|\partial B_r|}\int_{B_r}\beta_4[u](u-u^*)+\beta_2[u]\nabla_g^2(u-u^*)d\sigma+\alpha_2(u^*)\nabla_g^2u^*\#\nabla_g^2u^*,$$

$$\begin{aligned}III = {} & \frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_4[u](u-u^*)+\beta_2[u]\nabla_g^2(u-u^*)+\beta_3[u]\nabla_g(u-u^*)d\sigma\\ & +\alpha_3(u^*)\nabla_g^2u^*\#\nabla_g u^*\#\nabla_g u^*\end{aligned}$$

and

$$IV = \frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_4[u](u-u^*)+\beta_3[u]\nabla_g(u-u^*)d\sigma+\alpha_4(u^*)\nabla_g u^*\#\nabla_g u^*\#\nabla_g u^*\#\nabla_g u^*.$$

In summary, u^* satisfies an equation similar to (2.16) except an error term of the form

$$\frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_1[u]\nabla_g\triangle_g w+\beta_2[u]\nabla_g^2 w+\beta_3[u]\nabla_g w+\beta_4[u]wd\sigma.$$

Subtract the equation of u^* with (2.16) and handle the terms like $\alpha_1(u)\nabla_g\triangle_g u\#\nabla_g u - \alpha_1(u^*)\nabla_g\triangle_g u^*\#\nabla_g u^*$ as before to get

$$\begin{aligned}\triangle_g^2 w = {} & \beta_1'[u]\nabla_g\triangle_g w+\beta_2'[u]\nabla_g^2 w+\beta_3'[u]\nabla_g w+\beta_4'[u]w\\ & +\frac{1}{|\partial B_r|}\int_{\partial B_r}\beta_1[u]\nabla_g\triangle_g w+\beta_2[u]\nabla_g^2 w+\beta_3[u]\nabla_g w+\beta_4[u]wd\sigma+h,\end{aligned}$$

where

$$h=\tau(u)-\frac{1}{|\partial B_r|}\int_{\partial B_r}\tau(u)d\sigma.$$

To see that h satisfies (b) of Definition 1, we notice that $4(1-\frac{1}{p})>0$ and

$$\left\||x|^{4(1-\frac{1}{p})}h\right\|_{L^p(B_\delta\setminus B_{\lambda_i R})}\le \delta^{4(1-\frac{1}{p})}\|h\|_{L^p(B_\delta\setminus B_{\lambda_i R})}\le C\delta^{4(1-\frac{1}{p})}\|\tau(u_i)\|_{L^p(B_1)}.$$

Since $\tau(u_i)$ is uniformly bounded in L^p, the lemma follows by choosing δ small. □

Now we apply Theorem 4 to the function w_i.

Lemma 3. *For any $0<\varepsilon<\varepsilon_1$ and sufficiently small $\delta>0$, we have*

(2.17) $$F_l(w_i)\le C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(l-l_0)L}+e^{-\min\{8(1-1/p),1\}(l_i-l)L}\right),$$

for $l_0<l<l_i$.

Proof. Let the set of $l (l_0 < l < l_i)$, for which the condition (2.8) is not true, be denoted by $\{j_1, \cdots, j_{n_i}\}$ and we assume that

$$l_0 < j_1 < j_2 < \cdots < j_{n_i} < l_i.$$

By definition, for each $l = j_k$,

$$\max_{l-1,l,l+1} \||x|^{4(1-1/p)} h_i\|^2_{L^p(A_l)} \geq \eta_0 F_l(w_i). \tag{2.18}$$

Then we have

$$\begin{aligned} F_l(w_i) &\leq C \max_{l-1,l,l+1} \left\| |x|^{4(1-\frac{1}{p})} h_i \right\|_{L^p(A_l)} \\ &\leq Ce^{-8(1-1/p)lL} \\ &\leq C\delta^{8(1-1/p)} e^{-8(1-1/p)(l-l_0)L} \\ &\leq C\varepsilon^2 e^{-8(1-1/p)(l-l_0)L}, \end{aligned}$$

if we choose δ small.

By the choice of j_k, the condition (2.8) holds for $j_k < l < j_{k+1}$, $k = 1, ..., i-1$. By an application of Theorem 4 (see also Lemma 4.2 in [10]), we have, for $j_k < l < j_{k+1}$

$$\begin{aligned} F_l(w_i) &\leq C \left(e^{-L(l-j_k)} F_{j_k}(w_i) + e^{-L(j_{k+1}-l)} F_{j_{k+1}}(w_i) \right) \\ &\leq C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} \right). \end{aligned}$$

So, if $j_1 = l_0 + 1$ and $j_{n_i} = l_i - 1$, the inequality (2.17) follows immediately. If not, assuming $j_1 > l_0 + 1$, by Theorem 4 again, we have, for $l_0 < l < j_1$,

$$\begin{aligned} F_l(w_i) &\leq C \left(e^{-L(l-l_0)} F_{l_0}(w_i) + e^{-L(j_1-l)} F_{j_1}(w_i) \right) \\ &\leq C \left(e^{-L(l-l_0)} F_{l_0}(w_i) + \varepsilon^2 e^{-\min\{8(1-1/p),1\}(l-l_0)L} \right) \\ &\leq C\varepsilon^2 e^{-\min\{8(1-1/p),1\}(l-l_0)L}. \end{aligned}$$

Similarly, if $j_{n_i} < l_i - 1$, we have, for $j_{n_i} < l < l_i - 1$,

$$\begin{aligned} F_l(w_i) &\leq C \left(e^{-L(l-j_{n_i})} F_{j_{n_i}}(w_i) + e^{-L(l_i-l)} F_{l_i}(w_i) \right) \\ &\leq C \left(\varepsilon^2 e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-L(l_i-l)} F_{l_i}(w_i) \right) \\ &\leq C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-L(l_i-l)} \right) \\ &\leq C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-\min\{8(1-1/p),1\}(l_i-l)L} \right). \end{aligned}$$

□

Since w_i satisfies (2.4), we may use Lemma 1 to get estimates for the derivatives of w_i and the tangential derivatives of u_i. In the following, (r, θ) is the polar coordinates where $\theta \in S^3$ is a point of the unit sphere. A function $u(r, \theta)$ is also considered a function of $(\tilde{t}, \theta)$, where $r = e^{\tilde{t}}$. We denote the gradient operator on S^3 by ∇_{S^3} and the Laplacian on S^3 by $\triangle_{S^3}$.

Remark 9. Since we have only L^p norm of bi-tension fields bounded, we may not prove pointwise decay bound for tangential derivatives. Hence we need the following lemma as a replacement.

Lemma 4.

$$\begin{aligned}\int_{(-lL,-(l-1)L)\times S^3}&\left(|\triangle_{S^3}u_i|^2+|\partial_{\tilde t}\nabla_{S^3}u_i|^2\right)d\tilde t d\theta \\ \le\ & C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(l-l_0)L}+e^{-\min\{8(1-1/p),1\}(l_i-l)L}\right).\end{aligned}\tag{2.19}$$

Or equivalently,

$$\begin{aligned}\int_{[\tilde t,\tilde t+1]\times S^3}&\left(|\triangle_{S^3}u_i|^2+|\partial_{\tilde t}\nabla_{S^3}u_i|^2\right)d\tilde t d\theta \\ \le\ & C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(\log\delta-\tilde t)}+e^{-\min\{8(1-1/p),1\}(\tilde t-\log\lambda_i R)}\right).\end{aligned}$$

Proof. Setting

$$\tilde w(x)=w_i(e^{-(l-1)L}x),$$

we have

$$\begin{aligned}\|\tilde w\|^2_{L^2(A_0\cup A_1\cup A_2)}\ \le\ & C(F_{l-1}(w_i)+F_l(w_i)+F_{l+1}(w_i))\\ \le\ & C\varepsilon^2\left(e^{-\min\{8(1-1/p),1\}(l-l_0)L}+e^{-\min\{8(1-1/p),1\}(l_i-l)L}\right).\end{aligned}$$

By scaling, $\tilde w$ satisfies

$$\begin{aligned}\triangle_g^2\tilde w(r,\theta)=&\ \tilde a_1\nabla_g\triangle_g\tilde w+\tilde a_2\nabla_g^2\tilde w+\tilde a_3\nabla_g\tilde w+\tilde a_4\tilde w\\ &+\frac{1}{|\partial B_r|}\int_{\partial B_r}\tilde b_1\nabla_g\triangle_g\tilde w+\tilde b_2\nabla_g^2\tilde w+\tilde b_3\nabla_g\tilde w+\tilde b_4\tilde w d\theta+\tilde h(x).\end{aligned}\tag{2.20}$$

Here

$$\tilde h(x)=e^{-4(l-1)L}h(e^{-(l-1)L}x)$$

and

$$h(x)=\tau(u_i)-\frac{1}{|(\partial B_r)|}\int_{\partial B_r}\tau(u_i)d\sigma.$$

Letting $\lambda=e^{-(l-1)L}$, we have

$$\begin{aligned}&\left\|\tilde h\right\|_{L^p(A_0\cup A_1\cup A_2)}\\ =\ & \left(\int_{A_0\cup A_1\cup A_2}\left|\lambda^4h(\lambda x)\right|^p dx\right)^{\frac1p}\\ =\ & \lambda^{4(1-1/p)}(\int_{A_{l-1}\cup A_l\cup A_{l+1}}|h(x)|^p dx)^{1/p}\\ \le\ & Ce^{-4(1-1/p)(l-1)L}\\ \le\ & C\delta^{4(1-1/p)}e^{-4(1-1/p)(l-l_0)L}\\ \le\ & C\varepsilon e^{-4(1-1/p)(l-l_0)L},\end{aligned}\tag{2.21}$$

if δ is small.

Lemma 1 and the Sobolev embedding theorem imply that

$$\begin{aligned}
&\int_{(-lL,-(l-1)L)\times S^3} (|\triangle_{S^3} u_i|^2 + |\partial_{\tilde{t}} \nabla_{S^3} u_i|^2) d\tilde{t} d\theta \\
\leq\ & C\int_{A_1} (|\nabla^2 \tilde{w}|^2 + |\nabla \tilde{w}|^2) dx \\
\leq\ & C\varepsilon^2 \left(e^{-\min\{8(1-1/p),1\}(l-l_0)L} + e^{-\min\{8(1-1/p),1\}(l_i-l)L} \right).
\end{aligned}$$

□

2.3. **Proof of Theorem 2.** With the preparations of previous subsections, we may now prove Theorem 2. For the rest of the proof, we require $p \geq \frac{4}{3}$ and hence $\min\left\{8(1-\frac{1}{p}),1\right\} = 1$.

The rest of the proof is some type of Pohozaev argument. It follows the same line of Section 5 of [10]. However, the proof there made use of the explicit expression of bi-Laplace operator in polar coordinates of $\mathbb{R}^4$. Since we are now using the round metric on S^4, we think it is necessary to justify the reason why the proof still works. As it can be seen from below, this is not obvious and the proof depends on some detailed computation.

To begin with, we define a function (for $r < 1$)

$$t(r) = \int_1^r \frac{1}{\sin s} ds.$$

Obviously, $t'(r) = \frac{1}{\sin r}$. One may want to compare it with $\tilde{t}(r) = \log r$. In fact, we have

$$0 < \tilde{t}(r) - t(r) < C \quad \text{for} \quad r < 1$$

and $\tilde{t}'(r)$ is comparable with $t'(r)$. As a consequence, the result of Lemma 4 can be further rewritten as (noting that $p \geq 4/3$ here)

$$\begin{aligned}
&\int_{[t,t+1]\times S^3} \left(|\triangle_{S^3} u_i|^2 + |\partial_t \nabla_{S^3} u_i|^2\right) dt d\theta \\
\leq\ & C\varepsilon^2 \left(e^{-(t(\delta)-t)} + e^{-(t-t(\lambda_i R))}\right).
\end{aligned} \tag{2.22}$$

Recall that the metric is given by $g = dr^2 + \sin^2 r d\theta^2$. (Here $d\theta^2$ is the standard metric on the unit sphere.) To simplify the notations, we write $f(r) = \sin r$ and f' is the derivative of f with respect to r. The Laplace operator is

$$\triangle_g u = \partial_r^2 + \frac{3f'}{f}\partial_r u + \frac{1}{f^2}\triangle_{S^3} u.$$

By using $\partial_t = f\partial_r$, we may compute

$$\triangle_g u \quad = \quad f^{-2}\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right) u.$$

Writing $\triangle_g u = f^{-2}w$, we obtain

$$\begin{aligned}\triangle_g^2 u &= f^{-2}\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right)(f^{-2}w)\\ &= f^{-4}\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right)w\\ &\quad + f^{-2}\left(\partial_t^2(f^{-2})w + 2\partial_t(f^{-2})\partial_t w + 2f'\partial_t(f^{-2})w\right)\\ &= f^{-4}\left(\partial_t^2 - 2f'\partial_t + \triangle_{S^3}\right)w - 2\frac{f''}{f^3}w.\end{aligned}$$

By the definition of w and $f'' = -f$, we have

$$\begin{aligned}\triangle_g^2 u &= f^{-4}\left(\partial_t^2 - 2f'\partial_t + \triangle_{S^3}\right)\left(\partial_t^2 + 2f'\partial_t + \triangle_{S^3}\right)u + 2f^{-2}w\\ &= f^{-4}\left((\partial_t^2 + \triangle_{S^3})^2 - 4(f'\partial_t)(f'\partial_t)\right)u\\ &\quad + f^{-4}\left((\partial_t^2 + \triangle_{S^3})(2f'\partial_t) - (2f'\partial_t)(\partial_t^2 + \triangle_{S^3})\right)u + 2f^{-2}w.\end{aligned}$$

In comparison with the case of flat metric, f causes some extra terms. It is the primary goal here to show that we can handle these extra terms properly.

$$4(f'\partial_t)(f'\partial_t) = 4(f')^2\partial_t^2 - 4f^2 f'\partial_t,$$

where we have used $\partial_t = f'\partial_r$ and $f'' = -f$ because $f(r) = \sin r$.

Note that $\triangle_{S^3}$ commutes with $f'\partial_t$ and we compute

$$\begin{aligned}&\partial_t^2(2f'\partial_t) - (2f'\partial_t)\partial_t^2\\ =\ &\partial_t^2(2f')\partial_t + 2\partial_t(2f')\partial_t^2\\ =\ &-4f^2 f'\partial_t - 4f^2\partial_t^2.\end{aligned}$$

In summary, we have

$$\triangle_g^2 u = f^{-4}\left((\partial_t^2 + \triangle_{S^3})^2 - 4\partial_t^2\right)u + 2f^{-2}w, \tag{2.23}$$

where we have used $(f')^2 + f^2 = 1$.

Remark 10. The first term in the above formula is almost the same as the flat case. The importance of the computation is to show the error caused by the round metric is just $f^{-2}w$. Since w involves only first and second order derivatives, it can be controlled by the energy. *If* there is a third order derivative term here, then the proof below would fail.

By the definition of τ, we have

$$\int_{S^3} f^4\triangle_g^2 u\cdot\partial_t u d\theta = \int_{S^3} f^4\tau(u)\cdot\partial_t u d\theta.$$

By (2.23), the above is equivalent to

$$\int_{S^4}\left((\partial_t^2 + \triangle_{S^3})^2 - 4\partial_t^2\right)u\partial_t u d\theta = \int_{S^3}(f^4\tau(u) - 2f^2 w)\partial_t u d\theta.$$

The left hand side is now completely identical to the form which is dealt with in Section 5 of [10]. For simplicity, we set

$$\tilde\tau(u) = \tau(u) - 2f^{-2}w = \tau(u) - 2\triangle_g u.$$

Since u has finite energy, $\tilde\tau(u)$ is also uniformly bounded in L^p for $p\in[4/3,2]$.

The same computation as in [10] gives

$$
\begin{aligned}
(2.24) \qquad & \partial_t \int_{S^3} \partial_t u \partial_t^2 u d\theta - \int_{S^3} \frac{3}{2} \left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta \\
= & \int_{S^3} -\frac{1}{2}\left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 + \int_{-\infty}^{t} \int_{S^3} f^4 \tilde{\tau}(u) \cdot \partial_t u ds d\theta.
\end{aligned}
$$

We will integrate the above inequality from $t(\lambda_i R)$ to $t(\delta)$. We estimate the right hand side first. Thanks to (2.22), we have

$$
\int_{t(\lambda_i R)}^{t(\delta)} \int_{S^3} -\frac{1}{2}\left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 d\theta \leq C\varepsilon^2.
$$

Transforming back to $x-$coordinates by $\partial_t = f\partial_r$ and $d\sigma = f^3 d\theta$, we get

$$
\begin{aligned}
& \left| \int_{t(\lambda_i R)}^{t(\delta)} \int_{-\infty}^{\tilde{t}} \int_{S^3} f^4 \tilde{\tau}(u) \cdot \partial_t u d\theta ds d\tilde{t} \right| \\
\leq & \int_{t(\lambda_i R)}^{t(\delta)} \int_{B_{r(\tilde{t})}} \left|\tilde{\tau}(u)\right| \left|f\partial_r u\right| dx d\tilde{t} \\
\leq & \int_{\lambda_i R}^{\delta} \int_{B_r} \left|\tilde{\tau}(u)\right| \left|\nabla u\right| dx dr \\
\leq & C\delta \left\|\tilde{\tau}(u)\right\|_{L^{4/3}(B_1)} \left\|\nabla u\right\|_{L^4(B_1)}.
\end{aligned}
$$

In summary, the integration of (2.24) yields (by taking σ small with respect to ε)

$$
\begin{aligned}
(2.25) \qquad & \int_{t(\lambda_i R)}^{t(\delta)} \int_{S^3} \frac{3}{2}\left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta dt \\
\leq & C\left(\int_{\{t(\delta)\}\times S^3} |\partial_t u \partial_t^2 u| d\theta + \int_{\{t(\lambda_i R)\}\times S^3} |\partial_t u \partial_t^2 u| d\theta + \varepsilon^2 \right) \\
\leq & C\left\|\partial_t u\right\|_{L^2(\{t(\delta)\}\times S^3)} \left\|\partial_t^2 u\right\|_{L^2(\{t(\delta)\}\times S^3)} \\
& + C\left\|\partial_t u\right\|_{L^2(\{t(\lambda_i R)\}\times S^3)} \left\|\partial_t^2 u\right\|_{L^2(\{t(\lambda_i R)\}\times S^3)} + C\varepsilon^2 \\
\leq & C\varepsilon^2,
\end{aligned}
$$

where the last inequality comes from the (2.15) and Sobolev embedding and trace theorem. In fact, we have $W^{4,p}(\Omega)$ embeds into $W^{3,2}(\Omega)$, which in turn embeds into $W^{2,2}(\partial\Omega)$.

Remark 11. We remark that in fact, the argument above gives an independent proof of the energy identity in the blow up analysis of biharmonic maps with tension field in L^p for some $p \geq \frac{4}{3}$.

For some fixed $t_0 \in [t(\lambda_i R), t(\delta)]$, set

$$
F(t) = \int_{t_0 - t}^{t_0 + t} \int_{S^3} \frac{3}{2}\left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta dt.
$$

F is defined for $0 \le t \le \min\{t_0 - t(\lambda_i)R, t(\delta) - t_0\}$. Integrating (2.24) from $t_0 - t$ to $t_0 + t$, we obtain

$$\begin{aligned} F(t) \;\; \le \;\; & \frac{1}{2\sqrt{3}} \left(\int_{\{t_0-t\}\times S^3} + \int_{\{t_0+t\}\times S^3} \right) \frac{3}{2} \left|\partial_t^2 u\right|^2 + 2\left|\partial_t u\right|^2 d\theta \\ & + \int_{t_0-t}^{t_0+t} \left(\int_{S^3} -\frac{1}{2}\left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 d\theta + \int_{-\infty}^{\tilde{t}} \int_{S^3} f^4 \tilde{\tau}(u)\cdot \partial_t u ds d\theta \right) d\tilde{t}. \end{aligned}$$

With the help of (2.22), we can have

$$\int_{t_0-t}^{t_0+t} \int_{S^3} \frac{-1}{2}\left|\triangle_{S^3} u\right|^2 + \left|\partial_t \nabla_{S^3} u\right|^2 d\theta ds \le C\varepsilon^2 \left(e^{-(t(\delta)-t_0)} + e^{-(t_0 - t(\lambda_i R))} \right) e^t.$$

On the other hand,

$$\begin{aligned} & \left| \int_{t_0-t}^{t_0+t} \int_{-\infty}^{\tilde{t}} \int_{S^3} f^4 \tilde{\tau}(u)\cdot \partial_t u ds d\theta d\tilde{t} \right| \\ \le \;\; & \int_{t_0-t}^{t_0+t} \int_{B_{r(\tilde{t})}} \left|\tilde{\tau}(u)\right| \left|\nabla u\right| \left|f\right| dx d\tilde{t} \\ \le \;\; & C e^{\frac{1}{2}(t_0+t)} \left\|\tilde{\tau}(u)\right\|_{L^{4/3}(B_1)} \left\|\nabla u\right\|_{L^4(B_1)} \\ \le \;\; & C\delta^{1/2} e^{-1/2(\log\delta - t_0)} e^{t/2} \\ \le \;\; & C\delta^{1/2} e^{-1/2(t(\delta)-t_0)} e^{t} \end{aligned}$$

Remark 12. Note that since $r'(t) = \sin r$ and $\frac{1}{2}r \le \sin r \le r$ for $r < 1$, we have

$$e^t < r(t) < e^{t/2}$$

for $t < 0$.

Hence, if δ is small, we obtain

$$F(t) \le \frac{1}{2}\partial_t F(t) + C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right) e^t.$$

Multiplying e^{-2t} to both sides of the inequality, we have

$$(e^{-2t}F(t))' \ge -C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right) e^{-t}.$$

We assume without loss of generality that $t(\delta) - t_0 \le t_0 - t(\lambda_i R)$. Then, we integrate the above inequality from $t = 1$ to $t = t(\delta) - t_0$ to get

$$\begin{aligned} F(1) \;\; \le \;\; & e^{-2(t(\delta)-t_0)+2} F(t(\delta) - t_0) + C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right) \\ \le \;\; & C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right). \end{aligned}$$

Here we used (2.25).

Together with (2.22), we obtain

$$\int_{t_0-1}^{t_0+1} \int_{S^3} |\tilde{\nabla}^2 u|^2 + |\tilde{\nabla} u|^2 d\theta dt \le C\varepsilon^2 \left(e^{-\frac{1}{2}(t(\delta)-t_0)} + e^{-\frac{1}{2}(t_0 - t(\lambda_i R))} \right),$$

Here $\tilde{\nabla}$ is the gradient of $[t(\lambda_i R), t(\delta)] \times S^3$ with the product metric. Recall that $\left|\tilde{t}(r) - t(r)\right|$ is bounded by some universal constant and ∂_t and $\partial_{\tilde{t}}$ are comparable. Hence, we can translate the above decay estimate into a decay with respect to $\tilde{t} = \log r$.

$$\int_{\tilde{t}_0-1}^{\tilde{t}_0+1} \int_{S^3} |\tilde{\nabla}^2 u|^2 + |\tilde{\nabla} u|^2 d\theta d\tilde{t} \leq C\varepsilon^2 \left(e^{-\frac{1}{2}(\log(\delta) - \tilde{t}_0)} + e^{-\frac{1}{2}(\tilde{t}_0 - \log(\lambda_i R))} \right),$$

Direct computation shows that

$$\begin{aligned} \int_{B_{e^{\tilde{t}_0+1}} \setminus B_{e^{\tilde{t}_0-1}}} |\nabla^2 u|^2 + \frac{1}{|x|^2} |\nabla u|^2 dx &\leq C \int_{\tilde{t}_0-1}^{\tilde{t}_0+1} \int_{S^3} |\tilde{\nabla}^2 u|^2 + |\tilde{\nabla} u|^2 d\theta d\tilde{t} \\ &\leq C\varepsilon^2 \left(e^{-\frac{1}{2}(\log \delta - \tilde{t}_0)} + e^{-\frac{1}{2}(\tilde{t}_0 - \log \lambda_i R)} \right). \end{aligned}$$

Then by Sobolev embedding and the ε_0−regularity (Theorem 3), we have

$$\begin{aligned} & osc_{((\tilde{t}_0-1/2, \tilde{t}_0+1/2) \times S^3)} u \\ \leq\; & C\Big(\int_{B_{e^{\tilde{t}_0+1}} \setminus B_{e^{\tilde{t}_0-1}}} |\nabla^2 u|^2 + \frac{1}{|x|^2} |\nabla u|^2 dx\Big)^{1/2} + e^{4\tilde{t}_0(1-1/p)} \|\tau(u)\|_{L^p(B_{e^{\tilde{t}_0+1}} \setminus B_{e^{\tilde{t}_0-1}})} \\ \leq\; & C\varepsilon \left(e^{-\frac{1}{4}(\log \delta - \tilde{t}_0)} + e^{-\frac{1}{4}(\tilde{t}_0 - \log \lambda_i R)} \right). \end{aligned}$$

It is easy to derive the no neck estimate from here. Hence, we complete the proof of Theorem.

3. BOUNDING WIDTH BY ENERGY

Let N be the manifold in the Theorem 1 and g be any Riemannian metric on N. Since $N = N' \# T^m$, there is an embedded sphere S of dimension $m-1$ in N which separates N into N_1 and N_2 and N/N_2 is homeomorphic to N' and N/N_1 is homeomorphic to T^m. Here N/N_i is the quotient topology space by identifying all points in N_i as one point.

Let $\tilde{N}$ be a cover of N and $\tilde{g}$ be the lift of g. For a continuous map $u : S^4 \to N$, we define the width of u as

$$W(u) = \max_{x,y \in S^4} d_{(\tilde{N}, \tilde{g})}(\tilde{u}(x), \tilde{u}(y))$$

for a lift $\tilde{u}$ of u. Since the lift is unique up to the action of the deck transformation of $\tilde{N}$, the definition is independent of the choice of $\tilde{u}$.

Remark 13. It is perhaps more natural to use the universal cover. Theoretically, any cover will make the proof work. Since the main purpose is to construct examples, we use the definition which is convenient for our purpose. Of cause, the width depends on the choice of the cover.

Similarly, we can define the width of u from $\mathbb{R}^4$ to N by

$$W(u) = \sup_{x,y \in \mathbb{R}^4} d_{(\tilde{N}, \tilde{g})}(\tilde{u}(x), \tilde{u}(y))$$

for a lift $\tilde{u}$.

Remark 14. Since $\mathbb{R}^4$ is non-compact, it is possible that $W(u)$ is not finite. For application in this paper, we shall only be interested in the bubble map $u : \mathbb{R}^4 \to N$. There are several ways to see that for a bubble map with finite energy this width is finite. First, one can compose u with the stereographic projection and prove a removable singularity theorem for a PDE system similar but not identical to the biharmonic map equation as Wang did for quasi-biharmonic maps in Lemma 3.4 [16]. Second, the proof of removable singularity theorem in [10] can be applied in this case. Finally, since all such bubble maps come from the limit of some biharmonic map sequence, as remarked near the end of Section 2 of [10], this is a consequence of the main theorem in [10].

The main result of this section is

Lemma 5. *For any $C_1 > 0$, there is another constant C_2 depending on C_1 and the geometry of N such that any biharmonic map u from $\mathbb{R}^4$ (or S^4) to N with $\mathcal{E}(u) < C_1$ satisfies that $W(u) < C_2$.*

The proof uses the compactness properties of biharmonic maps (taking the bubbling into account). The non-compactness of $\mathbb{R}^4$ causes some technical problem. We need the following lemma to control the energy decay at the infinity.

Lemma 6. *There is a constant $\varepsilon_2 > 0$ depending on N. If $u : \mathbb{R}^4 \to N$ is a biharmonic map satisfying*

$$\int_{\mathbb{R}^4\setminus B_1} \left|\nabla^2 u\right|^2 + |\nabla u|^4 \, dx < \varepsilon_2,$$

then u is uniformly continuous at the infinity in the sense that for any $\varepsilon > 0$, there is $R > 0$ independent of u such that

$$osc_{\mathbb{R}^4\setminus B_R} u < \varepsilon.$$

Proof. The proof is just another version of Section 6 of [10]. The only difference is that for a removable singularity theorem, we study $B_1 \setminus \{0\}$, which is

$$B_1 \setminus \{0\} = \bigcup_{i=1}^{\infty} A_i$$

where

$$A_i = B_{e^{-(i-1)L}} \setminus B_{e^{-iL}},$$

while in this lemma, we study the asymptotic behavior of u on

$$\mathbb{R}^4 \setminus B_1 = \bigcup_{i=-\infty}^{0} A_i.$$

In the proof of the removable singularity theorem, we prove exponential decay as $i \to \infty$ ($|x| \to 0$), while here we prove exponential decay as $i \to -\infty$ ($|x| \to \infty$). We need ε_2 to be small, so that we can use Theorem 4 on $A_{i-1} \cup A_i \cup A_{i+1}$ for $i = -1, -2, \cdots$.

This lemma follows from the exponential decay of $|\nabla_{S^3} u|$ and $|\partial_t u|$. $\square$

Proof of Lemma 5. We only prove the case for $\mathbb{R}^4$ and the case for S^4 is simpler. Recall that the width of biharmonic map u from $\mathbb{R}^4$ is finite as discussed in Remark 14. If the lemma is not true, we can find a sequence of biharmonic maps $u_k : \mathbb{R}^4 \to N$ with $\mathcal{E}(u_k) \leq C_1$, but

$$\lim_{k\to\infty} W(u_k) = +\infty.$$

Since $\mathcal{E}(u_k)$ and $W(u_k)$ are invariant under the scaling, we may assume without loss of generality that

$$\int_{\mathbb{R}^4\setminus B_1} \left|\nabla^2 u_k\right|^2 + \left|\nabla u_k\right|^4 dx < \varepsilon_2. \tag{3.1}$$

(3.1) implies that the bubble points are restricted to $\bar{B}_1$.

Let u_∞ be the weak limit. Since there is no bubble outside $\bar{B}_1$, u_k converges to u_∞ on $B_R \setminus B_2$ uniformly for fixed R. Together with Lemma 6 and (3.1), the convergence is uniform on $\mathbb{R}^4 \setminus B_2$.

The bubbles are described as follows. Assume that there are l bubbles (including ghost bubbles, which is just trivial map), $\omega_i (i = 1, \cdots, l)$ and there are $m (m \le l)$ blow-up points $p_i (i = 1, \cdots, m)$ with $p_i \subset B_2$. Each ω_i is the limit of

$$w_{i,k}(x) := u_k(\lambda_{i,k}x + x_{i,k}).$$

Since there could be bubbles on top of ω_i, the convergence is strong on the domain

$$\Omega_{i,k} = B_R \setminus \left(\bigcup_s B_\delta(y_{k,s})\right),$$

where we use s to parameterize the bubbles on top of ω_i. Moreover, for each bubble ω_i, there is a neck region of the form $B_{r_2}(*) \setminus B_{r_1}(*)$, which we denote by $N_{i,k}$. There is no need to be precise about r_1, r_2 and the centers of the balls, it suffices to notice that the no neck theorem implies that

$$\lim_{k\to\infty} \operatorname{osc}_{N_{i,k}} u_k = o(\delta, R), \tag{3.2}$$

where $o(\delta, R)$ goes to zero when $\delta \to 0$ and $R \to \infty$.

By definition, if $\tilde{u}_k$ is a lift of u_k, we have

$$\begin{aligned} W(u_k) &= \sup_{y,z\in\mathbb{R}^4} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \\ &\le \sum_{i=1}^{l} \sup_{y,z\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k}y + x_{i,k}), \tilde{u}_k(\lambda_{i,k}z + x_{i,k})) \\ &\quad + \sup_{y,z\in N_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \\ &\quad + \sup_{y,z\in\mathbb{R}^4\setminus\bigcup_{i=1}^m B_\delta(p_i)} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)). \end{aligned} \tag{3.3}$$

Now we give an upper bound for the left hand side of the above equation. For the first line, since $w_{i,k}$ converges strongly to ω_i on $\Omega_{i,k}$, we have

$$\max_{y\in\Omega_{i,k}} d_{(N,g)}(w_{i,k}(y), \omega_i(y)) \le o(1).$$

Here $o(1)$ goes to zero as $k \to \infty$. Noticing that $\tilde{u}_k(\lambda_{i,k}x + x_{i,k})$ is a lift of $w_{i,k}(x)$ (defined on $\Omega_{i,k}$), we can find a lift of ω_i, denoted by $\tilde{w}_i$ such that

$$\max_{y\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k}y + x_{i,k}), \tilde{\omega}_i(y)) \le o(1).$$

Therefore, we have

$$\limsup_{k\to\infty} \sup_{y,z\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k}y + x_{i,k}), \tilde{u}_k(\lambda_{i,k}z + x_{i,k})) \le W(\omega_i). \tag{3.4}$$

For the second line, we need some general fact from Riemannian geometry as follows. There is some small $\sigma > 0$ depending on both (N, g) and $(\tilde{N}, \tilde{g})$ such that for any geodesic ball $B \subset N$ of radius σ and its lift $\tilde{B} \subset \tilde{N}$, we have that $(B, d_{(N,g)})$ is isometric to $(\tilde{B}, d_{(\tilde{N},\tilde{g})})$ as metric spaces.

Thanks to (3.2), for small δ and large R so that the image $u_k(N_{i,k})$ lies in a geodesic ball of radius σ, we have

$$\limsup_{k\to\infty} \sup_{y,z\in N_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \leq Co(\delta, R). \tag{3.5}$$

To bound the last line in (3.3), it suffices to note that u_k converges uniformly on $\mathbb{R}^4 \setminus \bigcup_m B_\delta(p_i)$ to u_∞. To see this, we note that u_k converges strongly on $B_2 \setminus \cup B_\sigma(p_i)$ and u_k converges strongly on $\mathbb{R}^4 \setminus B_2$ as remarked earlier. Hence,

$$\limsup_{k\to\infty} \sup_{y,z\in\mathbb{R}^4\setminus\bigcup_{i=1}^m B_\delta(p_i)} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \leq W(u_\infty). \tag{3.6}$$

(3.4), (3.5) and (3.6) add up to give an upper bound for $W(u_k)$, which contradicts the assumption that $\lim_{k\to\infty} W(u_k) = \infty$ and hence proves the lemma. □

4. Proof of the main theorem

Let $u(t)$ be a solution to (1.1) with $u(0) = u_0$. Along the flow,

$$\frac{d}{dt} E(u) \leq 0.$$

Hence, $E(u)$ is uniformly bounded (before the possible blow-up at least). Since the target manifold is compact, u is bounded and hence $\mathcal{E}(u)$ is also uniformly bounded.

The key observation to the proof is that for some $C_1 > 0$ and arbitrarily large C_3, we can choose u_0 with $\mathcal{E}(u_0) < C_1$ and any smooth u' homotopic to u_0 satisfies $W(u') > C_3$.

Assuming that such u_0 is found, we claim that $u(t)$ must blow-up in finite time and hence Theorem 1 is proved. If otherwise, the solution exists for any $t > 0$. Since

$$\int_0^\infty \int_{S^4} |\partial_t u|^2 \, dv dt < \infty,$$

we may choose a sequence of t_k going to ∞ such that

$$\lim_{k\to\infty} \|\partial_t u\|_{L^2}(t_k) \to 0.$$

For simplicity, we denote $u(t_k)$ by u_k.

Since $\mathcal{E}(u_k)$ is bounded and the $\varepsilon-$regularity (Theorem 3) holds, the usual blow-up analysis works. Assume that there are l bubbles $\omega_i (i = 1, \cdots, l)$, which is the limit of $u_k(\lambda_{i,k}x + x_{i,k})$ and $m(m < l)$ blow-up points p_i. Let $\Omega_{i,k}$ and $N_{i,k}$ as before. We still have

$$\begin{aligned} W(u_k) \leq & \sum_{i=1}^{l} \sup_{y,z\in\Omega_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(\lambda_{i,k}y + x_{i,k}), \tilde{u}_k(\lambda_{i,k}z + x_{i,k})) \\ & + \sup_{y,z\in N_{i,k}} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)) \\ & + \sup_{y,z\in\mathbb{R}^4\setminus\bigcup_{i=1}^m B_\delta(p_i)} d_{(\tilde{N},\tilde{g})}(\tilde{u}_k(y), \tilde{u}_k(z)). \end{aligned}$$

By Theorem 2, we can bound the right hand side by

$$\sum_{i=1}^{l} W(\omega_i) + W(u_\infty) + 1.$$

By Lemma 5, each $W(\omega_i)$ and $W(u_\infty)$ is bounded by a constant C_2 depending on C_1. Moreover, the number of bubbles is also bounded by a constant depending on C_1. Hence, there is a constant C_4 such that

$$\limsup_{k\to\infty} W(u_k) < C_4.$$

This would be a contradiction and hence proves Theorem 1 if $C_4 < C_3$.

Now let's show how to construct u_0.

Recall that $N = N'\#T^m$. There is a natural cover of N, which is obtained by modifying $\mathbb{R}^m$. $\mathbb{R}^m$ is the universal cover of T^m, with the deck transformation group $G = \mathbb{Z}^m$. Let p_0 be any point of $\mathbb{R}^m$ and let the orbit of the action of G containing p_0 be $\{p_i\}_{i=0}^{\infty}$. Suppose U_i be a small neighborhood of p_i diffeomorphic to the ball of dimension m and $V \subset N'$ be an open set diffeomorphic to a ball. For each $i = 0, 1, \cdots$, we remove U_i from $\mathbb{R}^m$ and identify the boundary of U_i with the boundary of a copy of $N' \setminus V$, which we denote by W_i. The new complete non-compact manifold is denoted by $\tilde{N}$. G acts on $\tilde{N}$ naturally and the quotient is N. If N is equipped with a Riemannian metric g and $\tilde{g}$ is the pull back metric, then the projection $\pi : \tilde{N} \to N$ is isometric map.

Since $\pi_4(N')$ is not trivial, there is a smooth map $h : S^4 \to N'$, which is not homotopic to constant map. Since $m > 4$ and h is not surjective, assume by deforming it smoothly that

(1) $h(S^4) \subset N' \setminus \bar{V}$;

(2) h maps the entire southern hemisphere to a single point $q \in N' \setminus \bar{V}$.

Let h_i be the copy of h from S^4 to W_i and q_i be the copy of q in W_i.

For any C_3, pick i such that

$$d_{(\tilde{N},\tilde{g})}(W_0, W_i) > C_3.$$

Let $\Psi(\Phi)$ be the stereoprojection from $\mathbb{R}^4$ to S^4, which maps the infinity to the south (north) pole and maps ∂B_1 to the equator. Consider the map $w : \mathbb{R}^4 \to W_i$ defined by

$$w(x) = h_i \circ \Psi(x).$$

w is a constant map outside B_1. Set

$$C_1 = E(w) + E(h) + 1.$$

We claim that for σ very small, we can find smooth u_0 satisfying

(1)

$$u_0 = \begin{cases} \pi \circ h_0(x) & x \in S^4 \setminus B_\sigma(S); \\ \pi \circ w(\frac{\Phi^{-1}(x)}{\sigma^2/2}) & x \in B_{\sigma^2}(S). \end{cases}$$

(2) $E(u_0) < C_1$.

By the above definition, we observe that $u_0|_{\partial B_\sigma(S)} = q_0$ and $u_0|_{\partial B_{\sigma^2}(S)} = q_i$. The first observation follows trivially from the definition of h_0. For the latter, we notice that $\left|\Phi^{-1}(x)\right|$ is almost σ^2 for every $x \in \partial B_{\sigma^2}(S)$, because Φ is almost an isometry near S and σ is going to be small.

Since the energy is scaling invariant and Φ is almost isometric in small neighborhood of the south pole, we have

$$\left(\int_{S^4\setminus B_\sigma(S)} + \int_{B_{\sigma^2}(S)}\right) |\triangle u_0|^2\, dv < E(h) + E(w) + \frac{1}{2}.$$

It suffices to show that we can define u_0 on $B_\sigma(S)\setminus B_{\sigma^2}(S)$ so that u_0 is smooth and the contribution to the energy on this part is smaller than $\frac{1}{2}$. By choosing σ small, the metric of S^4 on B_σ is close to the flat metric. Hence, it suffices to check this with flat metric.

Let $\gamma : [0,1] \to \tilde{N}$ be the shortest geodesic in $\tilde{N}$ connecting q_0 to q_i. Let $\varphi : [0,1] \to [0,1]$ be a smooth function satisfying

(1) $\varphi' \geq 0$;

(2) $\varphi(x) = 0$ for all $0 \leq x \leq \frac{1}{8}$ and $\varphi(x) = 1$ for all $\frac{7}{8} \leq x \leq 1$;

(3) $|\varphi'| + |\varphi''| \leq C$ for some universal constant C.

Set

$$u_0(x) = \pi\circ\gamma\circ\varphi\left(\frac{\log\sigma - \log|x|}{-\log\sigma}\right).$$

For simplicity, we write L for $d_{(\tilde{N},\tilde{g})}(q_0, q_i)$. Note that

$$|(\pi\circ\gamma)'| = L.$$

Since γ and $\pi\circ\gamma$ are geodesics, we have

$$(\pi\circ\gamma)'' + B(\pi\circ\gamma)((\pi\circ\gamma)', (\pi\circ\gamma)') = 0$$

where B is the second fundamental form of N. Therefore,

$$|(\pi\circ\gamma)''| = CL^2.$$

We estimate the derivative of u_0 as follows.

$$|\partial_r u_0| \leq \frac{CL}{r(-\log\sigma)}$$

and

$$\left|\partial_r^2 u_0\right| \leq \frac{CL}{r^2(-\log\sigma)}.$$

Hence,

$$\begin{aligned}
&\int_{B_\sigma\setminus B_{\sigma^2}} |\triangle u_0|^2\, dx \\
\leq\ & C\int_{\sigma^2}^{\sigma} \left|\partial_r^2 u_0 + \frac{3}{r}\partial_r u_0\right|^2 r^3 dr \\
\leq\ & \frac{CL^2}{(\log\sigma)^2}\int_{\sigma^2}^{\sigma} \frac{1}{r} dr \\
\leq\ & \frac{CL^2}{(-\log\sigma)}.
\end{aligned}$$

For any L, we can choose σ so that the above is as small as we want. Hence, we check that u_0 satisfies $E(u_0) < C_1$. It remains to check that for any map u' homotopic to u_0,

$W(u') > C_3$. Let $\tilde{u}'$ be the lift of u', which is homotopic to the following lift of u_0,

$$\tilde{u}_0 = \begin{cases} h_0(x) & x \in S^4 \setminus B_\sigma(S); \\ \gamma \circ \varphi(\frac{\log \sigma - \log|x|}{-\log \sigma}) & x \in B_\sigma(S) \setminus B_{\sigma^2}(S); \\ w(\frac{\Phi^{-1}(x)}{\sigma^2/2}) & x \in B_{\sigma^2}(S). \end{cases}$$

We claim that $\tilde{u'} \cap W_0 \neq \emptyset$ and $\tilde{u'} \cap W_i \neq \emptyset$. To see this, consider a continuous map $\tilde{\pi}$ from $\tilde{N}$ to N' (precisely, a manifold homeomorphic to N'), which maps any point in $\tilde{N} \setminus W_0$ to one point. If $\tilde{u}' \cap W_0$ is empty, then $\tilde{\pi} \circ \tilde{u}'$ is a constant map. However, $\tilde{\pi} \circ \tilde{u}_0$ is homotopic to h_0 and hence is nontrivial. The proof for $\tilde{u'} \cap W_i \neq \emptyset$ is the same.

In summary, we have constructed a map u_0 such that $E(u_0) < C_1$ and $W(u') > C_3$ for any u' homotopic to u_0. This finishes the proof of Theorem 1.

References

[1] Adams, R. and Fournier, J., *Sobolev spaces*, Second edition. Pure and Applied Mathematics (Amsterdam), 140. Elsevier/Academic Press, Amsterdam, 2003.

[2] Breiner, C. and Lamm, T., *Compactness results for sequences of approximate biharmonic maps*, Pacific J. Math., **276** (2015) 59-92.

[3] Chang, K.-C., Ding, W.-Y. and Ye, R., *Finite-time blow-up of the heat flow of harmonic maps from surfaces*, J. Differential Geom., **36(2)** (1992) 507-515.

[4] Chen, J.-Y. and Li, Y.-X., *Homotopy classes of harmonic maps of the stratified* 2*−spheres and applications to geometric flows*, Adv. Math., **263** (2014), 357-388.

[5] Cooper, M., *Critial $O(d)$-equivariant biharmonic map*, Calc. Var. PDE, **54** (2015), 2895-2919.

[6] Ding, W.-Y. and Tian G., *Energy identity for a class of approximate harmonic maps from surfaces*, Comm. Anal. Geom., **3** (1995), 543-554.

[7] Gastel, A., *The extrinsic polyharmonic map heat flow in critical dimensions*, Adv. Geom., **6(4)** (2006) 501-521.

[8] Lamm, T., *Heat flow for extrinsic biharmonic maps with small initial energy*, Ann. Global Anal. Geom., **26(4)** (2004) 369-384.

[9] Li, Y.-X. and Wang, Y.-D., *A counterexample to the energy identity for sequences of α-harmonic maps*, Pacific J. Math., **274** (2015) 107-123.

[10] Liu, L. and Yin, H., *Neck analysis of biharmonic maps*, arXiv:1312.4600.

[11] Qing, J. and Tian, G., *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math., **(50)** 1997 295-310.

[12] Simon, L., *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems.* Ann. Math., **118** (1983) 525-571.

[13] Struwe, M., *On the evolution of harmonic mappings of Riemannian surfaces*, Comment Math. Helv., **60** (1985) 558-581.

[14] Topping, P., *Winding behaviour of finite-time singularities of the harmonic map heat flow*, Math. Z., **247** (2004) 279-302.

[15] Wang, C.-Y., *Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold*, Math. Z., **247** (2004) 65-87.

[16] Wang, C.-Y., *Remarks on biharmonic maps into spheres*, Calc. Var. PDE, **21** (2004) 221-242.

[17] Wang, C.-Y., *Heat flow of biharmonic maps in dimension four and its applications*, Pure Appl. Math. Q., **3(2)** (2007) 595-613.

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China

E-mail address: LLEI1988@mail.ustc.edu.cn

Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Techology of China, Hefei, 230026, China

E-mail address: haoyin@ustc.edu.cn

JEPE Vol 1, 2015, p. 387-402

Orthogonal $\perp$

NEW GEOMETRIC ESTIMATES FOR EULER ELASTICA

JOHN MCCUAN

ABSTRACT. Elastic curves, whose curvature depends linearly on height, were studied by L. Euler. We generalize an elementary comparison theorem relating pairs of these curves.

1. INTRODUCTION

If a planar curve is given locally by the graph of a function $u = u(x)$, and the corresponding curvature satisfies

$$\frac{u''}{(1+u'^2)^{3/2}} = u, \tag{1.1}$$

we say the curve is an *elastic curve.* The following comparison result for elastic graphs is essentially due to R. Finn [Fin10].

Theorem 1. *If u and v are solutions of* (1.1) *on a common interval* $[a, b]$ *with* $u(a) < v(a)$ *and* $u'(a) = v'(a)$*, then*

$$u(x) < v(x) \quad \text{and} \quad u'(x) < v'(x) \quad \text{for } a < x \le b.$$

We wish to generalize Theorem 1 in several ways to allow for certain parametric elastic curves. The curvature condition defining elastic graphs readily generalizes to parametric curves, and it is customary to introduce an arclength parameter s and the *inclination angle* ψ along the curve defined by

$$\begin{cases} \dot{x}(s) = \cos\psi(s) \\ \dot{z}(s) = \sin\psi(s). \end{cases}$$

The curvature condition then becomes

$$\dot{\psi}(s) = z(s).$$

Under this rephrasing, notice that the initial conditions of Theorem 1 may be rewritten as

$$x_1(0) = x_2(0) = a, \quad z_1(0) < z_2(0), \quad \text{and} \quad \psi_1(0) = \psi_2(0)$$

where the common value $\psi_1(0) = \psi_2(0)$ lies between $-\pi/2$ and $\pi/2$. See Figure 1(left). We wish to allow this initial value for ψ to range also between $-3\pi/2$ and $-\pi/2$ as indicated in Figure 1(right). Notice that each of the curves indicated possesses a second point

2010 *Mathematics Subject Classification.* Primary: 53A04; Secondary: 49Q10.

Key words and phrases. Euler elastica.

Received 18/06/2015, accepted 12/11/2015.

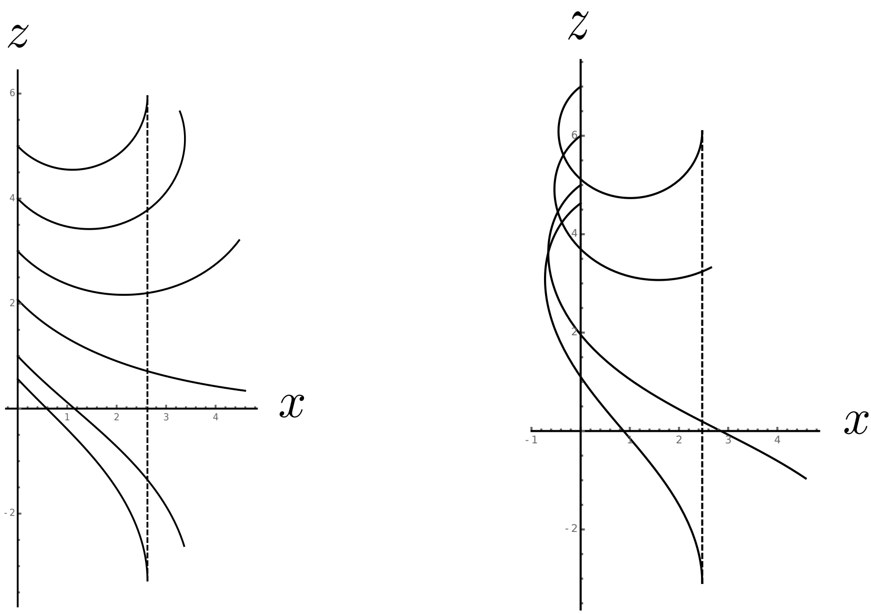


FIGURE 1. Elastic graphs (left) and parametric elastic curves (right)

$(x_j(\sigma_j), z_j(\sigma_j))$ with $x_j(\sigma_j) = a$ and $z_j(\sigma_j) < z_j(0)$. Among the conclusions of Theorem 2 below is that if $z_1(0) < z_2(0)$, then

$$z_1(\sigma_1) < z_2(\sigma_2) \quad \text{and} \quad \psi_1(\sigma_1) < \psi_2(\sigma_2). \tag{1.2}$$

Intuition might suggest that these assertions follow simply from the fact that "the curvature of the higher curve is greater at corresponding points." That the situation is more subtle and that this intuition, though correct in the non-parametric case, is actually incorrect in the parametric case may be illustrated by considering arcs of circles. Let $x_1(0) = x_2(0) = a$ with $\dot{\psi}_1 = z_1(0) < \dot{\psi}_2 = z_2(0)$ and $\psi_1(0) = \psi_2(0) \in (-\pi/2, \pi/2)$ so that (x_j, z_j) parameterizes a circular graph with curvature $z_j(0)$ for $j = 1, 2$ over some interval to the right of $x = a$. See Figure 2(left). We have then the conclusion of Theorem 1: If $s_1, s_2 > 0$ with $x_1(s_1) = x_2(s_2)$, then

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2). \tag{1.3}$$

If, on the other hand, the same conditions hold for arcs of circles with $\psi_1(0) = \psi_2(0) \in (-3\pi/2, -\pi/2)$, then one finds immediately that

$$\psi_1(\sigma_1) = \psi_2(\sigma_2)$$

where σ_j is the first positive arclength for which $x_j(\sigma_j) = a$, $j = 1, 2$. Thus, there is clearly something more involved in the parametric case of elastic curves.

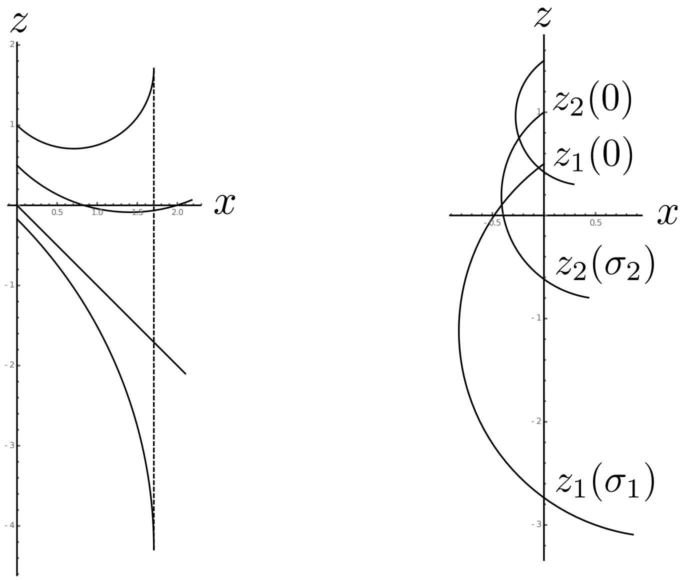


FIGURE 2. Circular graphs (left) and parametric circles (right)

2. Correspondence and comparison

We are primarily interested in solutions of

$$\begin{cases} \dot{x} = \cos\psi, & x(0) = a \\ \dot{z} = \sin\psi, & z(0) = \zeta \\ \dot{\psi} = z, & \psi(0) = \theta \end{cases} \tag{2.1}$$

for which $-3\pi/2 < \theta < -\pi/2$ and there is some first positive arclength $\sigma > 0$ for which $x(\sigma) = a$. We begin with a characterization of solutions for which this is the case. This will put us in a position to state a somewhat technical generalization of Theorem 1 to parametric elastic curves. First observe that the initial value problem

$$\begin{cases} \dot{z} = \sin\psi, & z(0) = \zeta \\ \dot{\psi} = z, & \psi(0) = \theta \end{cases}$$

decouples from the conditions on x, namely $\dot{x} = \cos\psi$ and $x(0) = a$, and admits a conserved quantity

$$h(\psi, z) = \frac{z^2}{2} + \cos\psi = \frac{\zeta^2}{2} + \cos\theta.$$

See Figure 3.

Lemma 1 (preliminary characterization of solutions)**.** *Given θ with $-3\pi/2 < \theta < -\pi/2$, there is a unique value $\zeta_- = \zeta_-(\theta) > 0$ for which solutions of (2.1) with $\zeta > \zeta_-$ are precisely those having*

$$\sup_{s>0} x(s) > a.$$

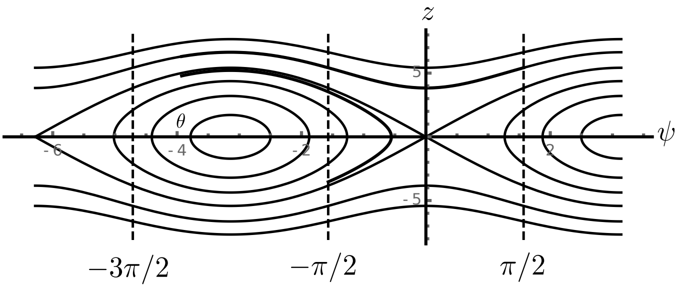


FIGURE 3. Phase plane diagram for ψ and z.

For each $\zeta \geq \zeta_-$ there is a unique first positive arclength $\ell = \ell(\theta, \zeta)$ for which $\psi(\ell) = -\pi/2$. Moreover, there is a unique value $\zeta_* = \zeta_*(\theta) > \zeta_-$ determined by

$$\frac{\zeta_*^2}{2} + \cos\theta = 1, \tag{2.2}$$

and we have

(1) *If $\zeta < \zeta_-$, then $x(s) < a$ for $s > 0$.*
(2) *If $\zeta_- \leq \zeta < \zeta_*$, then there is a unique second positive arclength $r = r(\theta, \zeta)$ for which the solution is vertical; $\psi(r) = -\pi/2$, and*
  (a) *$\dot{x}(s) = \cos\psi(s) < 0$ for $0 \leq s < \ell$, and $\dot{x}(s) = \cos\psi(s) > 0$ for $\ell \leq s < r$.*
  (b)
$$\lim_{\zeta \searrow \zeta_-} x(r(\theta, \zeta)) = a.$$
  (c)
$$\frac{d}{d\zeta} x(r(\theta, \zeta)) > 0, \quad \zeta_- < \zeta < \zeta_*.$$
  (d)
$$\lim_{\zeta \nearrow \zeta_*} x(r(\theta, \zeta)) = +\infty.$$
(3) *If $\zeta = \zeta_*$, then $\dot{x}(s) = \cos\psi(s) < 0$ for $0 \leq s < \ell$, and $\dot{x}(s) = \cos\psi(s) > 0$ for $\ell \leq s$ with*

$$\lim_{s \nearrow \infty} x(s) = +\infty. \tag{2.3}$$

(4) *If $\zeta > \zeta_*$, then there is a unique second positive $r = r(\theta, \zeta)$ for which the solution is vertical; $\psi(r) = \pi/2$, and*
  (a) *$\dot{x}(s) = \cos\psi(s) < 0$ for $0 \leq s < \ell$, and $\dot{x}(s) = \cos\psi(s) > 0$ for $\ell \leq s < r$.*
  (b)
$$\lim_{\zeta \searrow \zeta_*} x(r(\theta, \zeta)) = +\infty.$$
  (c)
$$\frac{d}{d\zeta} x(r(\theta, \zeta)) < 0, \quad \zeta > \zeta_*.$$
  (d)
$$\lim_{\zeta \nearrow \infty} x(r(\theta, \zeta)) = a.$$

Proof. The existence of ζ_* determined by (2.2) and the conserved quantity $z^2/2+\cos\psi = 1$ is immediate; we start the proof from this point. The existence and uniqueness of the values ℓ and r also follow from the phase diagram in Figure 3. Notice that ℓ and r will be well-defined as long as $\zeta \geq \sqrt{-2\cos\theta}$. We will see below that $\zeta_- > \sqrt{-2\cos\theta}$.

To see the limit (2.3) of the third assertion, note that $\dot\psi = z > 0$, so $x = x(s)$ may be expressed as a function $\xi = \xi(\psi)$ for $\theta \leq \psi < 0$ with

$$\frac{d\xi}{d\psi} = \frac{\dot x}{\dot\psi} = \frac{\cos\psi}{z} = \frac{\cos\psi}{\sqrt{2}\sqrt{1-\cos\psi}}$$

since $z^2/2+\cos\psi = 1$. Thus, observing that $\psi(s) \nearrow 0$ as $s \nearrow +\infty$,

$$\begin{aligned} x(s) = \xi(\psi) &= a + \frac{1}{\sqrt{2}}\int_\theta^\psi \frac{\cos t}{\sqrt{1-\cos t}}\,dt \\ &\geq a + \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/4} \frac{\cos t}{\sqrt{1-\cos t}}\,dt + \int_{-\pi/4}^{\psi} \frac{1}{|t|}\,dt\right) \\ &\to \infty \quad \text{as } \psi \to 0, \end{aligned}$$

and the third assertion is established. We next turn to the second assertion and consider $0 < \zeta < \zeta_*$. In this case, a maximum inclination angle $\psi_m \in (-\pi, 0)$ is determined by

$$\cos\psi_m = c := \frac{\zeta^2}{2} + \cos\theta \quad \text{so that} \quad \frac{dc}{d\zeta} = \zeta \quad \text{and} \quad \frac{d\psi_m}{d\zeta} = -\zeta\csc\psi_m > 0. \tag{2.4}$$

Note also in this case, the existence of a unique first positive arclength $m = m(\theta,\zeta) > \ell$ for which $\psi(m) = \psi_m$, and this is also the first positive arclength for which $z(m) = 0$.

If $0 < \zeta \leq \sqrt{-2\cos\theta}$, then $\psi_m \leq -\pi/2$ so $\dot x = \cos\psi \leq 0$ and $x(s) < a$ for $s > 0$. The same conclusion holds for $\zeta \leq 0$, so these cases comprise part of the first assertion. For $\sqrt{-2\cos\theta} < \zeta < \zeta_*$, we find

$$\begin{aligned} x(r) - a &= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + \int_{-\pi/2}^{\psi_m} \frac{\cos t}{\sqrt{c-\cos t}}\,dt - \int_{\psi_m}^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt\right) \\ &= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + 2\int_{-\pi/2}^{\psi_m} \frac{\cos t}{\sqrt{c-\cos t}}\,dt\right) \\ &= \frac{1}{\sqrt{2}}\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + 2\sqrt{2}\int_{-\pi/2}^{\psi_m} \csc^2 t\sqrt{c-\cos t}\,dt. \end{aligned}$$

In particular, it follows from the last expression for $x(r)$ that

$$\frac{d}{d\zeta}x(r) = \zeta\left(-\frac{1}{\sqrt{2}}\int_\theta^{-\pi/2} \frac{\cos t}{2(c-\cos t)^{3/2}}\,dt + \sqrt{2}\int_{-\pi/2}^{\psi_m} \frac{\csc^2 t}{\sqrt{c-\cos t}}\,dt\right) > 0 \tag{2.5}$$

since $\cos t < 0$ for $\theta \leq t < -\pi/2$. This establishes part (c) of the second assertion. Next, using (2.2) and (2.4) we can write

$$c = \frac{\zeta^2}{2} + 1 - \frac{\zeta_*^2}{2} = \cos\psi_m.$$

Thus, using a Taylor approximation of $c - \cos t$ at $t = \psi_m$, we obtain some small positive ϵ for which $0 < c - \cos t = \sin\psi_m(t-\psi_m) + \circ(|t-\psi_m|) < 2\sin\psi_m(t-\psi_m)$ for $\psi_m - \epsilon < t < \psi_m$.

Using the second expression for $x(r)$ and assuming $\psi_m - \epsilon > \pi/4$, we find

$$\begin{aligned}
x(r) - a &\geq \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + 2\int_{\psi_m - \epsilon}^{\psi_m} \frac{\cos \pi/4}{\sqrt{2\sin\psi_m (t - \psi_m)}}\, dt\right)\\
&= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + \frac{1}{\sqrt{|\sin\psi_m|}}\int_{\psi_m - \epsilon}^{\psi_m} \frac{1}{\sqrt{\psi_m - t}}\, dt\right)\\
&= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + 2\sqrt{\frac{\epsilon}{|\sin\psi_m|}}\right)\\
&\to x(\ell(\theta, \zeta_*); \zeta_*) - a + \infty \quad \text{as } \zeta \nearrow \zeta_*
\end{aligned}$$

since $\psi_m \nearrow 0$ as $\zeta \nearrow \zeta_*$ and $\epsilon > 0$ is fixed. This establishes part (d) of the second assertion. Part (b) of the second assertion and the rest of the first assertion now follow from the monotonicity (2.5) if we let ζ_- be the unique value of ζ for which $x(r(\theta, \zeta); \zeta) = a$.

Finally, we turn to the fourth and last assertion of the lemma. In this case ψ increases monotonically and

$$\begin{aligned}
x(r) - a &= \frac{1}{\sqrt{2}}\int_\theta^{\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt\\
&= \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt + \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt\right)\\
&= \frac{1}{\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{\sqrt{c + \cos \tau}}\, d\tau + \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{c - \cos t}}\, dt\right)\\
&\geq \frac{1}{\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{\sqrt{c + \cos \tau}}\, d\tau + \int_{-\pi/4}^{0} \frac{\cos \pi/4}{\sqrt{(\zeta^2 - \zeta_*^2)/2 + t^2/2}}\, dt\right)\\
&= \frac{1}{\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{\sqrt{c + \cos \tau}}\, d\tau + \sinh^{-1}\left(\frac{\pi/4}{\sqrt{\zeta^2 - \zeta_*^2}}\right)\right)\\
&\to x(\ell(\theta, \zeta_*); \zeta_*) - a + \infty \quad \text{as } \zeta \searrow \zeta_*.
\end{aligned}$$

This is part (b). To see the monotonicity of part (c), we return to the third expression for $x(r)$.

$$\begin{aligned}
\frac{d}{d\zeta}x(r) &= -\frac{\zeta}{2\sqrt{2}}\left(-\int_{\pi+\theta}^{\pi/2} \frac{\cos \tau}{(c + \cos \tau)^{3/2}}\, d\tau + \int_{-\pi/2}^{\pi/2} \frac{\cos t}{(c - \cos t)^{3/2}}\, dt\right)\\
&\leq -\frac{\zeta}{2\sqrt{2}}\int_{-\pi/2}^{-\pi+\theta} \frac{\cos \tau}{(c - \cos \tau)^{3/2}}\, d\tau\\
&< 0.
\end{aligned}$$

Finally, we return once again to the third expression for $x(r)$ and estimate directly to find

$$x(r) > a + \frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi+\theta} \frac{\cos t}{\sqrt{c - \cos t}}\, dt > a.$$

Furthermore,

$$\lim_{\zeta\nearrow\infty} x(r) \le a + \frac{1}{\sqrt{2}} \lim_{\zeta\nearrow\infty} \int_{-\pi/2}^{\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\, dt \le a + \lim_{\zeta\nearrow\infty} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{c}}\, dt = a + \pi \lim_{\zeta\nearrow\infty} \frac{1}{\sqrt{c}} = a.$$

This establishes the last part of the fourth assertion. □

In view of Lemma 1, our first comparison result may be stated as follows:

Theorem 2. *If (x_1, z_1, ψ_1) and (x_2, z_2, ψ_2) are solutions of (2.1) with $-3\pi/2 < \theta < -\pi/2$ and $\zeta = \zeta_1 = z_1(0)$ and $\zeta = \zeta_2 = z_2(0)$ respectively with $\zeta_- = \zeta_-(\theta) \le \zeta_1 < \zeta_2$, then for $j = 1, 2$, there is a unique first positive arclength σ_j for which $x_j(\sigma_j) = a$ and*

$$\{(x_j(s), z_j(s)) : 0 \le s \le \sigma_j\}$$

consists of an upper graph

$$\mathcal{U}_j = \{(x_j(s), z_j(s)) : 0 \le s \le \ell_j\}$$

where $\ell_j = \ell(\theta, \zeta_j)$ and a lower graph

$$\mathcal{L}_j = \{(x_j(s), z_j(s)) : \ell_j \le s \le \sigma_j\}.$$

The inclination angle ψ_j increases monotonically from θ to $-\pi/2$ with s for $0 \le s \le \ell_j$, and this provides a natural one-to-one correspondence between the upper graphs $\mathcal{U}_1$ and $\mathcal{U}_2$.
1. *Each upper graph may be parameterized by inclination angle $t \mapsto (\alpha_j(t), \beta_j(t)) = (x_j(s), z_j(s))$ for $t \in [\theta, -\pi/2]$ (and $s \in [0, \ell_j]$). Comparison of the points $p = (\alpha_1(t), \beta_1(t))$ and $q = (\alpha_2(t), \beta_2(t))$ in the upper graphs, as indicated in Figure 4(left), is given by*

$$\alpha_1(t) < \alpha_2(t) \quad \text{and} \quad \beta_1(t) < \beta_2(t) \quad \text{for } \theta < t \le -\pi/2. \tag{2.6}$$

A one-to-one correspondence between the lower graphs $\mathcal{L}_1$ and $\mathcal{L}_2$ is determined as follows:
2. *Let $B_j = B_j(\xi)$ for $x_j(\ell_j) \le \xi \le a$ express $\mathcal{L}_j$ as a graph over the horizontal x-axis. Two points $P = (\xi_1, B_1(\xi_1))$ and $Q = (\xi_2, B_2(\xi_2))$, in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, correspond to one another if for some $t \in [\theta, -\pi/2]$ (an inclination angle along the upper graphs) we have*

$$\alpha_1(t) = \xi_1 \quad \text{and} \quad \alpha_2(t) = \xi_2,$$

i.e., the x coordinates of the points on the lower graphs are shared with points on the upper graphs having the same inclination. This correspondence is illustrated in Figure 4(right). Under this correspondence, if $s_1 > \ell_1$ and $s_2 > \ell_2$ with

$$P = (x_1(s_1), z_1(s_1)) = (\xi_1, B_1(\xi_1)) \quad \text{and} \quad Q = (x_2(s_2), z_2(s_2)) = (\xi_2, B_2(\xi_2)),$$

then

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2). \tag{2.7}$$

The last assertion may be compared with (1.3). Among its corollaries is assertion (1.2) mentioned in the introduction which is obtained by taking $\xi_1 = \xi_2 = a$ and $t = \theta$.

In the next section we will prove the following generalization of Finn's result which allows us to concatenate the nonparametric comparison onto the parametric comparison.

Theorem 3. *If u and v are solutions of (1.1) on a common interval $[a, b]$ with $u(a) \le v(a)$ and $u'(a) \le v'(a)$, then either $u \equiv v$ or*

$$u(x) < v(x) \quad \text{and} \quad u'(x) < v'(x) \quad \text{for } a < x \le b.$$

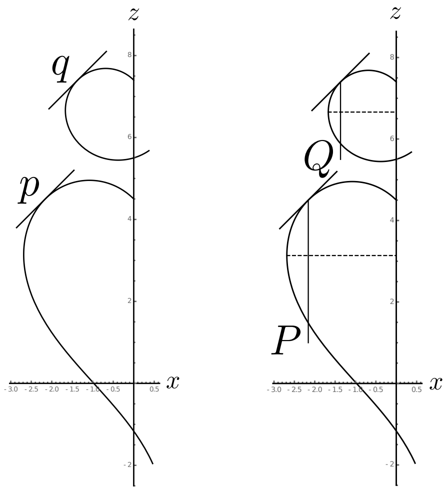


FIGURE 4. One-to-one correspondence of the upper graphs by inclination angle (left) and one-to-one correspondence of the lower graphs (right); the points P and Q in the lower graphs correspond because they share abscissas with p and q respectively which correspond in the upper graphs (by inclination angle).

Corollary 2. *If (x_1, z_1, ψ_1) and (x_2, z_2, ψ_2) are solutions of* (2.1) *with $-3\pi/2 < \theta < -\pi/2$ and $\zeta = \zeta_1 = z_1(0)$ and $\zeta = \zeta_2 = z_2(0)$ respectively with $\zeta_- = \zeta_-(\theta) \le \zeta_1 < \zeta_2$ as described in Theorem* 2*, and both solutions extend as graphs over some common interval $[a, b]$ on the horizontal axis so that $\sigma_j \le s_j$ for $j = 1, 2$ and $a \le x_1(s_1) = x_2(s_2) \le b$, then*

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2). \tag{2.8}$$

Remarks

(1) A minor modification of the proofs below allows the generalization of the curvature condition to $\dot{\psi} = \kappa z$ where κ is a positive constant.
(2) The results can also be generalized (with some minor technical changes) to allow the initial inclination angle $\theta = -3\pi/2$. It is probably possible to obtain some kind of generalization for $\theta < -3\pi/2$ as well, but that would require some fundamentally new ideas.
(3) The results for graphs, Theorem 1 and Theorem 3, can be generalized to graphs whose curvature is given by any strictly increasing function of height. Our proof below uses continuity of u and v on the closed interval $[a, b]$, but the values of the derivatives may be allowed to take the values $\pm\infty$ at the endpoints.

3. PROOFS

We begin with a proof of Theorem 3 (which also provides a proof of Theorem 1)
proof of Theorem 3: If $u(a) = v(a)$ and $u'(a) = v'(a)$, then $u \equiv v$.

In the complementary case, there is always some $\epsilon > 0$ for which $u(x) < v(x)$ and $u'(x) < v'(x)$ for $a < x < a + \epsilon$. Thus, taking

$$x_* = \min\{x > a : u(x) \geq v(x) \text{ or } u'(x) \geq v'(x)\},$$

assuming such a point exists, we can set $c = \max\{v(x) - u(x) : a \leq x \leq x_*\}$ and $w = u + c$. We see that w is a smooth function with $w \geq v$, and equality holds for at least one point $x_b \in (a, x_*]$. Also $w'(x_b) = v'(x_b)$. In a neighborhood of $x_b \in (a, x_*]$ we have $w \geq v$ but the graph of w has curvature strictly less than that of the graph of v. This leads to a contradiction. □

Proof of the upper graph comparison of Theorem 2:

We begin with a general comparison which applies to any continuous correspondence by inclination angle starting at $t = \theta$ with $\beta_1(\theta) < \beta_2(\theta)$.

Lemma 3. $\beta_1(t) < \beta_2(t)$.

Proof. If for some $t_0 > \theta$, we have $\beta_1(t_0) = \beta_2(t_0)$, then the horizontal translation of the first/lower curve parameterized by

$$t \mapsto (\alpha_1(t) + \alpha_2(t_0) - \alpha_1(t_0), \beta_1(t))$$

is an elastic curve which agrees with the second/upper elastic curve up to first order at $(\alpha_2(t_0), \beta_2(t_0))$. It follows that these two elastica are identical. In particular, $\beta_1(\theta) = \beta_2(\theta)$, which is a contradiction. □

On an upper graph, $\cos t \leq 0$ and

$$\alpha_j'(t) = \frac{\dot{x}_j}{\dot{\psi}_j} = \frac{\cos t}{\beta_j(t)}.$$

Thus, $\alpha_1'(t) \leq \alpha_2'(t) < 0$ with strict inequality except for $t = -\pi/2$. Thus, $\alpha_1(t) < \alpha_2(t)$ for $\theta < t \leq -\pi/2$, and this completes the first part of the proof of Theorem 2.

Proof of the lower graph comparison of Theorem 2:

As might be expected, this is rather more complicated than the first part. However, we have most of the framework set up in the statements of Lemma 1 and the result itself. As a matter of technical convenience, we rename the points P, Q, p, and q as P_1, P_2, P_3, and P_4 respectively so that $P_j = (x_1(s_j), z_1(s_j))$ for j odd and $P_j = (x_2(s_j), z_2(s_j))$ for j even. For the lower graph comparison it is natural to use $\xi = \xi_2 \in [x_2(\ell_2), a]$ as a parameter; it will be noted that each of the arclengths s_j, $j = 1, 2, 3, 4$ is uniquely and continuously determined as a function of ξ, and the dependence is smooth for $x_2(\ell_2) < \xi < a$. Consequently, we can attempt a direct comparison of the derivatives of the quantities in (2.7) with respect to ξ.

Lemma 4 (direct differentiation)**.** *The arclengths s_1 and s_2 are increasing functions of ξ and s_3 and s_4 are decreasing functions of ξ for $x_2(\ell_2) < \xi < a$ with*

(1)
$$\frac{d}{d\xi} z_1(s_1) = \tan \psi_1(s_1) \frac{z_2(s_4)}{z_1(s_3)} \quad \text{and} \quad \frac{d}{d\xi} z_2(s_2) = \tan \psi_2(s_2).$$

(2)
$$\frac{d}{d\xi} \psi_1(s_1) = \frac{z_1(s_1)}{\cos \psi_1(s_1)} \frac{z_2(s_4)}{z_1(s_3)} \quad \text{and} \quad \frac{d}{d\xi} \psi_2(s_2) = \frac{z_2(s_2)}{\cos \psi_2(s_2)}$$

(3) *If* $-\pi/2 < \psi_1(s_1) < 0$ *and* $\psi_1(s_1) < \psi_2(s_2) < \pi/2$, *then*

$$\frac{d}{d\xi} z_1(s_1) < \frac{d}{d\xi} z_2(s_2).$$

(4) *If*
   (a) $z_1(s_1) < 0$,
   (b) $z_1(s_1) < z_2(s_2)$,
   (c) $-\pi/2 < \psi_1(s_1) < 0$, *and*
   (d) $\psi_1(s_1) < \psi_2(s_2)$,

   then

$$\frac{d}{d\xi} \psi_1(s_1) < \frac{d}{d\xi} \psi_2(s_2).$$

Proof. Differentiating directly,

$$\frac{d}{d\xi} z_1(s_1) = \dot{z}_1(s_1) \frac{ds_1}{d\xi} = \sin \psi_1(s_1) \frac{ds_1}{d\xi}.$$

Also, since $x_1(s_1) = x_1(s_3)$ and $\psi_1(s_3) = \psi_2(s_4)$,

$$\cos \psi_1(s_1) \frac{ds_1}{d\xi} = \cos \psi_1(s_3) \frac{ds_3}{d\xi} \quad \text{and} \quad z_1(s_3) \frac{ds_3}{d\xi} = z_2(s_4) \frac{ds_4}{d\xi}.$$

Finally, since $x_2(s_4) = \xi$, we have

$$\frac{ds_4}{d\xi} = \frac{1}{\cos \psi_2(s_4)} \quad \text{and} \quad \frac{d}{d\xi} z_1(s_1) = \tan \psi_1(s_1) \frac{\cos \psi_1(s_3)}{\cos \psi_2(s_4)} \frac{z_2(s_4)}{z_1(s_3)} = \tan \psi_1(s_1) \frac{z_2(s_4)}{z_1(s_3)}.$$

The other differentiation formulas follow similarly.

Since we know from the comparison of the upper graphs that $z_2(s_4) > z_1(s_3)$ and $\tan \psi$ is increasing for $-\pi/2 < \psi < \pi/2$, the inequality of the third assertion is immediate.

For the last assertion, we note that under the hypotheses $\cos \psi_1(s_1) > 0$, and $\cos \psi$ is increasing for $-\pi/2 < \psi < 0$. If $z_2(s_2) < 0$ then $\psi_2(s_2) < 0$, and the inequality follows immediately using the monotonicity of $\cos \psi$. If $z_2(s_2) \geq 0$, then the inequality follows simply because the quantity on the left is negative and the quantity on the right is non-negative. □

Starting at the left-most points, i.e., for $\xi = x_2(\ell_2)$ where $\beta_1(-\pi/2) < \beta_2(-\pi/2)$, we know $z_1(s_1) < z_2(s_2)$ at least initially simply by continuity. Unfortunately, $\psi_2(\ell_2) = -\pi/2 = \psi_1(\ell_1)$, and we do not have the hypotheses of the fourth assertion in order to get started with the desired inequality between $\psi_1(s_1)$ and $\psi_2(s_2)$. Thus, an immediate application of the direct differentiation lemma seems problematic.

We turn instead to a secondary comparison which applies to portions of the lower graphs in one-to-one correspondence by inclination angle.

For each elastic curve, there is some maximal interval $[\ell_j, m_j^*)$ on which $\dot{\psi}_j = z_j > 0$. In fact, if for $\zeta_j^- < \zeta_j < \zeta_*$ we let $m_j = m_j(\theta, \zeta_j)$ denote the first positive arclength m for which $z_j(m) = 0$ and $\psi_j(m) = \psi_m(\theta, \zeta_j)$, then

$$m_j^* = \begin{cases} \min\{m_j, \sigma_j\} & \text{if } \zeta_j^- \leq \zeta_j < \zeta_*(\theta) \\ \sigma_j & \text{if } \zeta_j \geq \zeta_*(\theta). \end{cases}$$

Thus, we see the parameterization $t \mapsto (\alpha_j(t), \beta_j(t))$ by inclination angle is still valid for $\theta \leq t \leq \psi_j(m_j^*)$, and a one-to-one correspondence of the lower graphs is also possible for

$-\pi/2 \leq t \leq T = \min\{\psi_1(m_1^*), \psi_2(m_2^*)\}$. Lemma 3 still applies on this interval, and we know $\beta_1(t) < \beta_2(t)$.

Now we make the secondary comparison mentioned above.

Lemma 5. *Let $B_j^+ = B_j^+(\xi)$ for $x_j(\ell_j) \leq \xi \leq a$ express $\mathcal{U}_j$ as a graph over the horizontal x-axis. For each t with $-\pi/2 \leq t \leq T = \min\{\psi_1(m_1^*), \psi_2(m_2^*)\}$ there are corresponding points $\hat{P}_1 = (\alpha_1(t), \beta_1(t)) \in \mathcal{L}_1$ and $P_2 = (\alpha_2(t), \beta_2(t)) \in \mathcal{L}_2$. The second point $P_2 = (\alpha_2(t), \beta_2(t))$ shares its x-coordinate with a unique point*

$$P_4 = (\xi, B_2^+(\xi)) = (\alpha_2(t), B_2^+(\alpha_2(t))) = (x_2(s_4), z_2(s_4)) \in \mathcal{U}_2.$$

The inclination angle $\psi_2(s_4)$ for $0 \leq s_4 \leq \ell_2$ determines the point

$$P_3 = (\alpha_1(\psi_2(s_4)), \beta_1(\psi_2(s_4))) \in \mathcal{U}_1$$

by the inclination angle correspondence. Under these conditions

$$\alpha_1(\psi_2(s_4(t))) < \alpha_1(t) \quad \text{for } -\pi/2 < t \leq T. \tag{3.1}$$

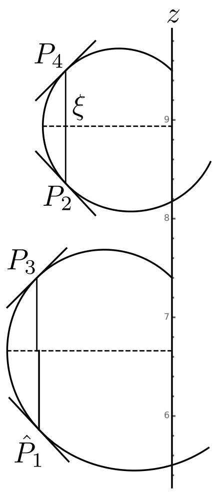


FIGURE 5. Secondary comparison: $\hat{P}_1$ and P_2 correspond by inclination angle; P_3 and P_4 correspond by inclination angle. If P_2 and P_4 have the same x-coordinate, then P_3 is to the left of $\hat{P}_1$.

Proof. An alternative phrasing of the assertion (3.1) is as follows: Each $\xi \in (x_2(\ell_2), \beta_2(T))$, determines points

$$P_2 = (\xi, B_2(\xi)) = (x_2(s_2), z_2(s_2)) \in \mathcal{L}_2 \quad \text{and} \quad P_4 = (\xi, B_2^+(\xi)) = (x_2(s_4), z_2(s_4)) \in \mathcal{U}_2.$$

Each of these points corresponds by inclination angle to a unique point in the first/lower elastic curve:

$$\hat{P}_1 = (x_1(\hat{s}_1), z_1(\hat{s}_1)) \in \mathcal{L}_1 \quad \text{where } \ell_1 < \hat{s}_1 \leq \psi_1^{-1}(T) \text{ and } \psi_1(\hat{s}_1) = \psi_2(s_2),$$

and

$$P_3 = (x_1(s_3), z_1(s_3)) \in \mathcal{U}_1 \quad \text{where } 0 \leq s_3 < \ell_1 \text{ and } \psi_1(s_3) = \psi_2(s_4). \tag{3.2}$$

Assertion (3.1) then says

$$x_1(s_3) < x_1(\hat{s}_1).$$

The arclength $\hat{s}_1$ is also a continuous function of $\xi \in [x_2(\ell_2), \alpha_2(T)]$ which is smooth on the interior of the interval. (Note that $\hat{s}_1 = s_1$ when $\xi = x_2(\ell_2)$; in this case $(x_1(s_1), z_1(s_1))$ and $(\xi, \alpha_2(-\pi/2))$ are the left-most points of each curve. Generally, $P_1 = (x_1(s_1), z_1(s_1))$ is a different point which lies directly below P_3.)

Differentiating as in the direct differentiation lemma, we find

$$\frac{d}{d\xi} x_1(s_3) = \dot{x}_1(s_3) \frac{ds_3}{d\xi} = \frac{\cos\psi_1(s_3)}{\cos\psi_2(s_4)} \frac{z_2(s_4)}{z_1(s_3)} = \frac{z_2(s_4)}{z_1(s_3)}.$$

Similarly,

$$\frac{d}{d\xi} x_1(\hat{s}_1) = \frac{z_2(s_2)}{z_1(\hat{s}_1)}.$$

We see then that the result will follow if we can show

$$\frac{z_2(s_4)}{z_1(s_3)} < \frac{z_2(s_2)}{z_1(\hat{s}_1)}. \tag{3.3}$$

Given the correspondence of $\hat{P}_1$ to P_2 and P_3 to P_4 by inclination angle t, we compute

$$\frac{d}{dt}\frac{\beta_2(t)}{\beta_1(t)} = \frac{\sin t}{\beta_1^2}\left(\frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1}\right)$$

since $d\beta_j/dt = \dot{z}_j/\dot{\psi}_j = \sin t/\beta_j$. It follows that

$$\frac{d}{dt}\frac{\beta_2(t)}{\beta_1(t)} > 0 \quad \text{for } \max\{-\pi, \psi_2(s_4)\} < t < \min\{0, T\}.$$

We conclude

$$\frac{z_2(s_4)}{z_1(s_3)} < \frac{z_2(\ell_2)}{z_1(\ell_1)} < \frac{z_2(s_2)}{z_1(\hat{s}_1)} \quad \text{for } x_2(\ell_2) < \xi \le \min\{\alpha_2(T), \alpha_2(-\pi)\}. \tag{3.4}$$

Note: It may be that $-\pi < \theta$, but if so, then $\alpha_2(-\pi)$ is still well-defined and $\alpha_2(T) < \alpha_2(-\pi)$. One may also be concerned that $T > 0$; this can only happen if $\zeta_2 > \zeta_*$, and in this case, i.e., in case 4 of Lemma 1, we find

$$\begin{aligned}
\alpha_2(T) &= a + \frac{1}{\sqrt{2}}\left(\int_\theta^{-\pi} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + \int_{-\pi}^{-\pi/2} \frac{\cos t}{\sqrt{c-\cos t}}\,dt \right. \\
&\qquad\qquad \left. + \int_{-\pi/2}^{0} \frac{\cos t}{\sqrt{c-\cos t}}\,dt + \int_0^T \frac{\cos t}{\sqrt{c-\cos t}}\,dt\right) \\
&= \alpha_2(-\pi) + \frac{1}{\sqrt{2}}\left(\int_{-\pi/2}^{0}\left(\frac{\cos t}{\sqrt{c-\cos t}} - \frac{\cos t}{\sqrt{c+\cos t}}\right)dt + \int_0^T \frac{\cos t}{\sqrt{c-\cos t}}\,dt\right) \\
&> \alpha_2(0) \\
&> \alpha_2(-\pi).
\end{aligned}$$

(See the proof of Lemma 1.)

It remains to establish (3.3) when we know $\min\{\alpha_2(T), \alpha_2(-\pi)\} = \alpha_2(-\pi) < \alpha_2(T)$ and $\psi_2(s_4(T)) < -\pi$. For each t with $-3\pi/2 < \theta \le \psi_2(s_4(T)) \le t < -\pi$, and corresponding points

$$P_3 = (x_1(s_3), z_1(s_3)) = (\alpha_1(t), \beta_1(t)) \in \mathcal{U}_1$$

and

$$P_4 = (x_2(s_4), z_2(s_4)) = (\alpha_2(t), \beta_2(t)) \in \mathcal{U}_2,$$

there are symmetric points

$$\tilde{P}_3 = (\alpha_1(-2\pi - t), \beta_1(-2\pi - t)) = (\alpha_1(-2\pi - t), \beta_1(t)) \in \mathcal{U}_1$$

and

$$\tilde{P}_4 = (\alpha_2(-2\pi - t), \beta_2(-2\pi - t)) = (\alpha_2(-2\pi - t), \beta_2(t)) \in \mathcal{U}_2.$$

Since $-\pi < -2\pi - t < -\pi/2$ and the z-coordinates are the same, we have from (3.4) that

$$\frac{z_2(s_4)}{z_1(s_3)} = \frac{\beta_2(t)}{\beta_1(t)} < \frac{z_2(\ell_2)}{z_1(\ell_1)}.$$

The same kind of symmetry applies to the lower graphs with respect to $t = 0$ so that the quotient inequalities of (3.4) hold also for $x_2(\ell_2) < \xi \le \alpha_2(T)$. Thus, (3.3) follows and the lemma is proved.

As a technical note, we mention that the lower bound on s_3 in (3.2) can be given somewhat more precisely as follows. Consider the unique arclength $\mu_4 < \ell_2$ for which $x_2(\mu_4) = \alpha_2(T)$. Consider also the unique arclength $\mu_3 < \ell_1$ for which $\psi_1(\mu_3) = \psi_2(\mu_4)$. Then it is clear that $s_3 \ge \mu_3$. □

We are now in a position to determine how the correspondence of the lower graphs by inclination angle comes to an end.

Corollary 6. $T = \min\{\psi_1(m_1^*), \psi_2(m_2^*)\} = \psi_1(m_1^*) < \psi_2(m_2^*)$.

Proof. Notice that m_j^* is the first positive arclength s for which $(x_j(s), z_j(s))$ is not in the open quadrant $\mathcal{R} = \{(x, z) : x < a, z > 0\}$.

Assume $T = \psi_2(m_2^*) \le \psi_1(m_1^*)$. This means there is some $\hat{s}_1 \le m_1^*$ for which $P_2 = (x_2(s_2), z_2(s_2)) \in \partial\mathcal{R}$ and $\hat{P}_1 = (x_1(\hat{s}_1), z_1(\hat{s}_1)) \in \bar{\mathcal{R}}$ as well.

By Lemma 3, we know that $z_1(\hat{s}_1) < z_2(s_2)$. Since $z_1(\hat{s}_1) \ge 0$, this means $z_2(s_2) > 0$, and $x_2(s_2) = a$. But then $\xi = \xi_2 = \alpha_2(T) = a$. In particular, $x_2(s_4) = a$ and $\psi_2(s_4) = \theta$. This means $x_1(s_3) = a = x_1(\hat{s}_1)$ as well, but this contradicts Lemma 5 which says $x_1(s_3) < x_1(\hat{s}_1)$. □

Let $\mu_2^* < m_2^*$ be defined by

$$(x_2(\mu_2^*), z_2(\mu_2^*)) = (\alpha_2(T), \beta_2(T)).$$

Corollary 7. *If $P = (\xi_1, B_1(\xi_1))$ and $Q = (\xi_2, B_2(\xi_2))$ correspond as in Theorem 2, and $x_2(\ell_2) \le \xi_2 \le x_2(\mu_2^*)$, then*

$$z_1(s_1) < z_2(s_2) \quad \text{and} \quad \psi_1(s_1) < \psi_2(s_2)$$

as asserted in Theorem 2.

Proof. Comparing Figures 4 and 5, we see the identifications $Q = P_2$ and $q = (\xi_2, B_2^+(\xi_2)) = P_4$, and $P_3 = p = (x_1(s_3), z_1(s_3))$ from the proof of Theorem 2 are consistent with the secondary comparison of Lemma 5. Thus, the point $P = P_1$ is directly below $p = P_3$ on the lower graph $\mathcal{L}_1$ while $\hat{P}_1 = (x_1(\hat{s}_1), z_1(\hat{s}_1)) = (\alpha_1(\psi_2(s_2)), \beta_1(\psi_2(s_2)))$ is the point described in the secondary comparison. By Lemma 5,

$$\xi_1 = x_1(s_1) = x_1(s_3) < x_1(\hat{s}_1) = \alpha_1(\psi_2(s_2)).$$

Since x_1 and ψ_1 are both increasing for $\ell_1 \le s \le m_1^*$ and $\ell_1 < s_1, \hat{s}_1 \le m_1^*$, this means $s_1 < \hat{s}_1$ and

$$\psi_1(s_1) < \psi_1(\hat{s}_1) = \psi_2(s_2). \tag{3.5}$$

Thus, we have the angle comparison of (2.7) at least for $x_2(\ell_2) < \xi \le x_2(\mu_2^*)$.

In view of (3.5), we can apply part 3 of the direct differentiation lemma (Lemma 4) to conclude

$$z_1(s_1) < z_2(s_2)$$

as long as $-\pi/2 < \psi_1(s_1) \le 0$.

If there are further points for which $0 < \psi_1(s_1) < \psi_2(s_2)$, then since $s_1 < \hat{s}_1$ we have

$$z_1(s_1) < z_1(\hat{s}_1) < z_2(s_2)$$

by Lemma 3. This completes the proof of the corollary. □

We next consider the possibilities when $\xi = \xi_2 > x_2(\mu_2^*)$. We know $(x_1(m_1^*), z_1(m_1^*)) \in \partial\mathcal{R}$, and we consider various cases.

case 1. $x_1(m_1^*) = a$ and $z_1(m_1^*) \ge 0$.

In this case, $m_1^* = \sigma_1$. Let us assume that the closed set

$$F = \{s_2 \in (\mu_2^*, m_2^*] : z_1(s_1) \ge z_2(s_2) \text{ or } \psi_1(s_1) \ge \psi_2(s_2)\}$$

is nonempty. Then there is a first arclength $s = s_2^F \in F$ and a corresponding s_1^F where either $z_1(s_1^F) = z_2(s_2^F)$, $\psi_1(s_1^F) = \psi_2(s_2^F)$, or both.

case 1a. $x_1(m_1^*) = x_1(\sigma_1) = a$, $z_1(m_1^*) = z_1(\sigma_1) \ge 0$, and $\psi_1(\sigma_1) \le 0$.

In this case $\psi_1(s_1) < 0$ for $\ell_1 \le s_1 < \sigma_1$, and part 3 of the direct differentiation lemma (Lemma 4) applies for $\ell_1 < s_1 < s_1^F$ so that

$$z_1(s_1) < z_2(s_2) \quad \text{for } x_2(\ell_2) \le \xi \le x_2(s_2^F). \tag{3.6}$$

This means, in particular, that $z_2(s_2) > 0$ for $\mu_2^* \le s_2 \le s_2^F$. Since $\dot{\psi}_2 = z_2$, it follows that

$$\psi_2(s_2) \ge \psi_2(\mu_2^*) = \psi_1(\sigma_1) > \psi_1(s_1) \quad \text{for } x_2(\ell_2) \le \xi \le x_2(s_2^F). \tag{3.7}$$

Together (3.6) and (3.7) contradict the existence of $s_2^F = \min F$.

case 1b. $x_1(m_1^*) = x_1(\sigma_1) = a$, $z_1(m_1^*) = z_1(\sigma_1) \ge 0$, and $\psi_1(\sigma_1) > 0$.

In this case we know $\zeta_1 > \zeta_*$ and $z_1 > 0$ globally. Since $\zeta_2 > \zeta_1$, the same assertion holds for the second/upper curve. It follows that

$$\psi_2(s_2) > \psi_1(\sigma_1) > \psi_1(s_1) \quad \text{for } x_2(\mu_2^*) \le \xi \le x_2(s_2^F) \le a, \tag{3.8}$$

and

$$z_2(s_2) > z_1(\sigma_1) \quad \text{for } x_2(\mu_2^*) \le \xi_2 \le a.$$

Since the desired angle comparison of (2.7) is not violated, we must have

$$z_1(s_1^F) = z_2(s_2^F) > z_1(\sigma_1).$$

The only way this can happen is if $\psi_1(s_1^F) < 0$, i.e., $Q = P_2^F = (x_2(s_2^F), z_2(s_2^F))$ is to the right of the global minimum of the second/upper curve and $P_1^F = (x_1(s_1^F), z_1(s_1^F))$ is to the left of the global minimum on the first/lower curve. In summary, letting s_j^0, $j = 1, 2$ denote the arclength for which the respective minima are achieved, we have

$$s_2^0 < s_2^F \leq m_2^* = \sigma_2, \quad \ell_1 < s_1^F < s_1^0, \quad \text{and} \quad z_1(s_1^F) = z_2(s_2^F).$$

It follows that for $\epsilon > 0$ small enough, the point $\tilde{Q} = \tilde{P}_2 = (x_2(s_2^F - \epsilon), z_2(s_2^F - \epsilon))$ has

$$z_2(s_2^F - \epsilon) < z_2(s_2^F),$$

while the corresponding point $\tilde{P} = (x_1(s_1^\epsilon), z_1(s_1^\epsilon))$ for some $s_1^\epsilon < s_1^F$ satisfies

$$z_1(s_1^\epsilon) > z_1(s_1^F) = z_2(s_2^F).$$

Since $\xi^\epsilon = x_2(s_2^F - \epsilon) < x_2(s_2^F)$, this contradicts the fact that $s_2^F = \min F$.

These contradictions show that $F = 0$ and the conclusion of Theorem 2 must hold in case 1.

case 2. $x_1(m_1^*) < a$ and $z_1(m_1^*) = 0$.

We begin with $z_2(\mu_2^*) > 0 = z_1(m_1^*) = z_1(m_1)$ and $\psi_1(m_1^*) = \psi_1(m_1) < \psi_2(\mu_2^*)$. It will be noted that the hypotheses of parts 3 and 4 of the direct differentiation lemma hold initially for each $\xi > x_2(\mu_1^*)$. These hypotheses continue to hold and imply (2.7) for $x_2(\mu_1^*) \leq \xi \leq a$. □

In view of the foregoing proof, Corollary 2 may be sharpened/extended. In order to apply the comparison of graphs in Theorem 3 on an interval extending to the left of $x = a$ it is only necessary to have an initial $\xi \in [x_2(\ell_2), a]$ for which

$$B_1(\xi) \leq B_2(\xi) \quad \text{and} \quad B_1'(\xi) \leq B_2'(\xi).$$

These conditions will hold for $\xi = x_2(\mu_2^*)$ the termination point on the upper curve for the inclination angle correspondence. Furthermore, $B_1(x_2(\mu_2^*)) < B_2(x_2(\mu_2^*))$ as we now explain.

In the situation of **case 1** we know $x_1(s_1) = x_3(s_3) < x_2(s_4) = x_2(s_2) < x_1(\sigma_1) = a$ and $z_1(s_1) < z_2(s_2)$ for $x_2(\ell_2) \leq \xi \leq x_2(\mu_2^*)$. But also, $z_1(\hat{s}_1) < z_2(s_2)$ for $x_2(\ell_2) \leq \xi \leq x_2(\mu_2^*)$ by Lemma 3. Letting s_1^* denote the value of s_1 when $\xi = x_2(\mu_2^*)$, this means

$$x_1(s_1^*) < x_2(\mu_2^*) < x_1(\sigma_1) = a \quad \text{and} \quad B_2(x_2(\mu_2^*)) > \max\{B_1(x_1(s_1^*)), B_1(x_1(\sigma_1))\}.$$

By convexity, $B_2(x_2(\mu_2^*)) > B_1(x_2(\mu_2^*))$. Also, since $\psi_2(\mu_2^*) = \psi_1(\sigma_1)$ in this case, we have $\psi_2(\mu_2^*) > \psi_1(x_1^{-1}(B_1(x_2(\mu_2^*))))$ where x_1^{-1} is the inverse of the restriction of x_1 to $[\ell_1, \sigma_1]$. This implies $B_2'(x_2(\mu_2^*)) > B_1'(x_2(\mu_2^*))$.

In the situation of **case 2**, we again let s_1^* be the value of s_1 when $\xi = x_2(\mu_2^*)$, and we find

$$B_2(x_2(\mu_2^*)) > 0 = z_1(m_1) \quad \text{and} \quad B_2(x_2(\mu_2^*)) > z_1(s_1^*)$$

while

$$\psi_2(\mu_2^*) = \psi_2(m_1).$$

If $x_1(s_1^*) < x_2(\mu_2^*) \leq x_1(m_1)$, then the convexity argument we used in the situation of **case1** applies.

If $x_1(m_1) \leq x_2(\mu_2^*) < a$, then it is immediate that $B_2(x_2(\mu_2^*)) > 0 \geq B_1(x_2(\mu_2^*))$ and $B_2'(x_2(\mu_2^*)) = B_1'(x_1(m_1)) \geq B_1'(x_2(\mu_2^*))$.

Final remark/summary: We have considered three different kinds of comparisons between pairs of elastic curves. The first (Theorem 2 part 1 and Lemma 5) is a simple comparison by inclination angle which applies for some continuous intervals $0 \leq s \leq m_1^*$ and $0 \leq s \leq \mu_2^*$ when $\psi_j(0) = \theta \in (-3\pi/2, -\pi/2)$. The second (Theorem 2) is a somewhat complicated combination of inclination angle correspondence (on the upper graphs) and vertical correspondence by shared x-coordinates. The third (Theorems 1 and 3) is simple comparison of graphs over the x-axis. The three corresponding ranges of application overlap with the first comparison extending into the lower graphs (Lemma 5), the second applying precisely to the lower graphs, and the third applying at least where the first ceases to apply.

References

[Eul44] L. Euler. *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti*. Bosquet, Lausanne, 1744.

[Fin10] R. Finn. On Young's paradox, and the attractions of immersed parallel plates. *Physics of Fluids*, 22(017103), 2010.

School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332

E-mail address: mccuan@math.gatech.edu

JEPE Vol 1, 2015, p. 403-418

Orthogonal $\perp$

AN EXTRAPOLATION THEOREM IN NON-EUCLIDEAN GEOMETRIES AND ITS APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

JONAS SAUER

Abstract. We prove a generalization of an extrapolation theorem in the fashion of García-Cuerva and Rubio de Francia towards $\mathcal{R}$-boundedness on weighted Lebesgue spaces over locally compact abelian groups. This result can be applied to show maximal L^p regularity for differential operators that correspond to parabolic evolution equations subject to more general spatial geometries, for example the partially periodic Stokes operator. As a main tool, we generalize the classical Muckenhoupt theorem on maximal operators to locally compact abelian groups.

1. Introduction

In the setup of $\mathbb{R}^n$ the concept of Muckenhoupt weights has been studied extensively throughout the last four decades or so, with many remarkable results in the fields of harmonic analysis, weighted inequalities and partial differential equations (cf. [2], [9], [11], [12], [13], [14], [19], [20], [25]). For $q \in (1, \infty)$, a nonnegative weight function $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in the Muckenhoupt class $A_q(\mathbb{R}^n)$ if

$$\mathcal{A}_q(\omega) := \sup_{r>0} \sup_{y\in\mathbb{R}^n} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} \omega \,\mathrm{d}x \right) \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} \omega^{-\frac{q'}{q}} \,\mathrm{d}x \right)^{\frac{q}{q'}} < \infty,$$

where $B_r(x)$ denotes the open ball of radius r around the center x, and where q' is the Hölder conjugate of q. The weight ω is said to be in $A_1(\mathbb{R}^n)$ if there is a constant $c > 0$ such that $\mathcal{M}_{\mathbb{R}^n}\omega(x) \leq c\omega(x)$ for almost all $x \in \mathbb{R}^n$. Here, $\mathcal{M}_{\mathbb{R}^n}$ denotes the usual (centered) Hardy-Littlewood maximal operator on $\mathbb{R}^n$. These classes of weights have been introduced by Muckenhoupt, who considered such weights for bounded intervals and products of intervals [21].

Muckenhoupt weights are known to possess several interesting properties. In particular, the maximal operator is bounded on weighted L^q-spaces for $q \in (1, \infty)$, see Theorem V.3.1 in [25]. This result was used by García-Cuerva and Rubio de Francia to show their Extrapolation Theorem in Section IV.5 of [13], which states that if a family of operators is uniformly bounded in $L^q_\omega(\mathbb{R}^n)$ for one $q \in [1, \infty)$ but all $\omega \in A_q(\mathbb{R}^n)$, then it is already

2010 *Mathematics Subject Classification.* 42B25; 43A15; 43A70; 76D07.

Key words and phrases. Muckenhoupt weights; extrapolation; locally compact abelian groups; maximal operator.

Received 24/11/2015, accepted 26/01/2016.

The author has been supported by the DFG and JSPS as a member of the International Research Training Group Darmstadt-Tokyo IRTG 1529.

bounded in $L^p_\nu(\mathbb{R}^n)$ for *all* $p \in (1,\infty)$ and all $\nu \in A_p(\mathbb{R}^n)$. Strengthening this result towards $\mathcal{R}$-boundedness of families of operators as defined in Section 4, Fröhlich [12] proved maximal L^p-regularity of the Stokes operator on weighted spaces $L^q_\omega(\Omega)$, where Ω is the whole space, the half space or a bounded domain of class $C^{1,1}$. For details about maximal L^p-regularity, see e.g. [6], [17].

In this paper, we wish to generalize the theory of Muckenhoupt weights and extrapolation towards locally compact abelian groups G. We apply the abstract methods obtained here in [23], [24] in order to obtain maximal regularity of the partially periodic Stokes operator and to treat a spatially periodic nonlinear model describing the dynamics of nematic liquid crystal flows. Let us briefly discuss how the results of the present paper affect the Stokes equations in $\mathbb{R}^n$ exhibiting a periodic behaviour in some of the space dimensions. For simplicity, assume that there is exactly one direction of periodicity, say in x_n-direction. That is, we consider for a fixed time $T > 0$ the set of equations

$$\left\{\begin{array}{rll} \partial_t u - \Delta u + \nabla \mathfrak{p} & = f & \text{in } (0,T)\times\mathbb{R}^n, \\ \operatorname{div} u & = 0 & \text{in } (0,T)\times\mathbb{R}^n, \\ u(0) & = u_0 & \text{in } \mathbb{R}^n, \\ u(t,x',x_n+2\pi) & = u(t,x',x_n), & \\ \lim\limits_{|x'|\to\infty} u(t,x',x_n) & = 0, & \end{array}\right. \tag{1.1}$$

with periodic external force $f(t,x',x_n) = f(t,x',x_n+2\pi)$. We analyze this problem in an L^q-setting, whence the decay condition in the last line of (1.1) will always be fulfilled, at least in the sense of summability. Taking the work of Kyed [18] on the time-periodic Stokes equations as an inspiration and introducing the locally compact abelian group $G := \mathbb{R}^{n-1}\times\mathbb{R}/2\pi\mathbb{Z}$, we can use the Fourier transform on G to define the Helmholtz projection $\mathbb{P} : L^q(G) \to L^q_\sigma(G)$, $1 < q < \infty$, via

$$\mathcal{F}_G[\mathbb{P}f](\eta) := \left(I - \frac{\eta\otimes\eta}{|\eta|^2}\right)\mathcal{F}_G[f](\eta), \qquad 0 \neq \eta \in \hat{G} := \mathbb{R}^{n-1}\times\mathbb{Z},$$

and equivalently reformulate (1.1) as an abstract Cauchy problem on the Banach space $L^q_\sigma(G)$ via

$$\left\{\begin{array}{rll} \partial_t u + Au & = f & \text{in } (0,T), \\ u(0) & = u_0, & \end{array}\right. \tag{1.2}$$

where the partially periodic Stokes operator A is defined as usual, *i.e.*, $A := -\mathbb{P}\Delta$. We wish to establish maximal L^p-regularity of the partially periodic Stokes operator, which is equivalent to the $\mathcal{R}$-boundedness of the family of resolvent operators $\{\lambda(\lambda+A)^{-1} | \lambda \in \Sigma_{\vartheta+\frac{\pi}{2}}\}$ with $\Sigma_{\vartheta+\frac{\pi}{2}} := \{\lambda\in\mathbb{C} : |\arg\lambda| < \vartheta + \frac{\pi}{2}, \lambda \neq 0\}$ for some $\vartheta \in (0,\pi/2)$. Thus, the question is how to obtain the corresponding $\mathcal{R}$-bounds. In [24] we show that at least uniform bounds can be obtained via Fourier analysis on the group G. These uniform bounds hold true not only in $L^q(G)$, but in fact in all Muckenhoupt weighted $L^q_\omega(G)$-spaces as introduced in Section 3 below. Therefore, the aim of this paper is to establish an extrapolation theorem which ensures that $\mathcal{R}$-boundedness follows already from uniform boundedness in all weighted spaces. Theorem 2 gives exactly this.

Note that on locally compact abelian groups one can define a nontrivial, translation invariant, regular measure μ, called *Haar measure* [1], [4], [15], [27], with $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, such a measure is unique up to multiplication with a constant. However, we often deal with the measure $\mathrm{d}\mu_\omega := \omega\,\mathrm{d}\mu$, which is not translation invariant anymore. Therefore, if not stated otherwise, we shall drop the translation-invariance condition on μ. For $q \in [1,\infty]$ one can thus introduce the space $L^q(G)$ of q-integrable functions $f : G \to \mathbb{R}$, which turns into a Banach space if equipped with the usual norm

$$\|f\|_q := \left(\int_G |f|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}, \qquad q \in [1,\infty),$$
$$\|f\|_\infty := \mu\text{-}\operatorname*{ess\,sup}_G |f|.$$

Further we introduce the notion $L^{q,\infty}(G)$ for the weak $L^q(G)$-space, as introduced e.g. in [26]. Note that the space of continuous functions with compact support $C_0(G)$ is dense in $L^q(G)$ for all $q \in [1,\infty)$, see Appendix E.8 of [22] for details.

As we wish to carry over as many concepts known from classical harmonic analysis as possible to the general setting, we will have to assume that the group G is furnished with something that resembles the concept of balls and that the measure μ enjoys a doubling property with respect to these balls. We therefore make the following assumption.

Assumption 1. Suppose that G is a locally compact abelian group equipped with a nontrivial and regular measure μ, such that $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, assume that there is a local base of $0 \in G$ consisting of relatively compact measurable neighbourhoods U_k, $k \in \mathbb{Z}$, such that

(i) $\bigcup_{k\in\mathbb{Z}} U_k = G$,

(ii) $U_k \subset U_m$, if $k \le m$,

(iii) there exist a positive constant A and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $x \in G$

$$k < \theta(k),$$
$$U_k - U_k \subset U_{\theta(k)},$$
$$\mu(x + U_{\theta(k)}) \le A\mu(x + U_k).$$

Observe that necessarily $A \ge 1$ because $U_k \subset U_{\theta(k)}$.

Remark 1. From now on, we will always assume that the locally compact abelian group G admits a family of sets $(U_k)_{k\in\mathbb{Z}}$ satisfying Assumption 1. We will call any set of the form $x+U_k$, $x \in G$, $k \in \mathbb{Z}$ a *base set*. It is instructive to think of such base sets as an equivalent of balls in the $\mathbb{R}^n$ with center in x and radius 2^k. Observe that by the following considerations we can assume without loss of generality the base sets to be symmetric and the function θ to be non-decreasing.

(i) Replace θ by $\tilde\theta$ defined via

$$\tilde\theta(k) := \min\{l \in \mathbb{Z} : l > k \text{ with } U_k - U_k \subset U_l\}.$$

The thusly defined function is non-decreasing and satisfies $\tilde\theta(k) \le \theta(k)$ for all $k \in \mathbb{Z}$. Therefore, for all $x \in G$ and $k \in \mathbb{Z}$,

$$\mu(U_{\tilde\theta(k)}) \le \mu(U_{\theta(k)}) \le A\mu(U_k),$$

and hence we may assume that θ is non-decreasing.

(ii) We call a set $U \subset G$ *symmetric* if $U = -U$. Since G is abelian, the set $V := U - U$ is symmetric for any $U \subset G$. Replacing the base sets U_k by the symmetric sets $V_k := U_k - U_k$ and replacing the doubling constant A by A^2, we may assume that all of our base sets are symmetric. Indeed, the V_k still form a local base of $0 \in G$ consisting of relatively compact neighbourhoods, see e.g. Appendix B.4 of [22]. The inclusion $V_k \subset V_m$ for $k \leq m$ is obvious and the union of the $V_k (\supset U_k)$ covers the whole group.

Concerning condition (iii) of Assumption 1, we see

$$V_k - V_k \subset U_{\theta(k)} - U_{\theta(k)} = V_{\theta(k)}.$$

Moreover, the doubling property will be fulfilled with constant A^2, since for all $x \in G$ and all $k \in \mathbb{Z}$

$$\mu(x + V_{\theta(k)}) \leq \mu(x + U_{\theta^2(k)}) \leq A^2 \mu(x + U_k) \leq A^2 \mu(x + V_k).$$

Thus, from now on we will assume the base sets U_k to be symmetric and we will write $U_k - U_k = U_k + U_k =: 2U_k$.

Remark 2. Among the most prominent groups satisfying Assumption 1 are the groups $\mathbb{R}$, $\mathbb{Z}$, the torus $\mathbb{T}$ and finite products of these groups.

(i) In the case of the real numbers $\mathbb{R}$ equipped with the Lebesgue measure, define $U_k := (-2^{k-1}, 2^{k-1})$, $A = 2$ and $\theta(k) = k + 1$.

(ii) For integers, an analogous construction to (i) corresponding to the counting measure satisfies Assumption 1. Namely, choose $U_k := (-2^{k-1}, 2^{k-1}) \cap \mathbb{Z}$, $A = 3$ and $\theta(k) = k + 1$.

(iii) If one chooses the arc length as a measure on the torus, possible choices are $U_k := \{z \in \mathbb{C} : |\arg z| < 2^k\}$, $A = 2$ and $\theta = k + 1$.

Moreover, if G is a locally compact abelian group with an increasing sequence $(U_k)_{k \in \mathbb{Z}}$ of compact open subgroups, such that

$$\bigcup_{k \in \mathbb{Z}} U_k = G, \qquad \bigcap_{k \in \mathbb{Z}} U_k = \{0\},$$

then Assumption 1 is fulfilled if and only if $A := \sup_{k \in \mathbb{Z}} |U_k : U_{k-1}| < \infty$, and one may take $\theta(k) = k + 1$ in that case. See Examples 2.1.3 in [10] for details.

Let us define the (centered) *maximal operator* on G. Suppose $f \in L^1_{\mathrm{loc}}(G)$ and define the sublinear operator

$$\mathcal{M}_G f(x) := \sup_{k \in \mathbb{Z}} \frac{1}{\mu(x + U_k)} \int_{x + U_k} |f| \, \mathrm{d}\mu. \tag{1.3}$$

Note that $\mathcal{M}_G f$ is obviously lower semi-continuous and therefore measurable.

Our two main theorems can be viewed as direct generalizations of their equivalents in the classical $\mathbb{R}^n$-setup. For the definition of $A_q(G)$-consistency see Section 3.

Theorem 1. *Let G be a locally compact abelian group satisfying Assumption 1 and assume $q \in (1, \infty)$ and $\omega \in A_q(G)$. Then $\mathcal{M}_G$ is bounded in $L^q_\omega(G)$ with an $A_q(G)$-consistent bound.*

Theorem 2. *Let G be a locally compact abelian group satisfying Assumption 1. Suppose that $r, q \in (1, \infty)$, $\omega \in A_q(G)$ and that $\Omega \subset G$ is measurable. Moreover, assume that $\mathcal{T}$ is a* *family of linear operators such that for all* $\upsilon \in A_r(G)$ *there is an* $A_r(G)$*-consistent constant* $c_r = c_r(\upsilon) > 0$ *with*

$$\|Tf\|_{L^r_\upsilon(\Omega)} \le c_r \|f\|_{L^r_\upsilon(\Omega)}$$

for all $f \in L^r_\upsilon(\Omega)$ *and all* $T \in \mathcal{T}$*. Then every* $T \in \mathcal{T}$ *can be extended to* $L^q_\omega(\Omega)$ *and* $\mathcal{T}$ *is* $\mathcal{R}$*-bounded in* $\mathcal{L}(L^q_\omega(\Omega))$ *in the sense of Definition 4 with an* $A_q(G)$*-consistent* $\mathcal{R}$*-bound* c_q*.*

This paper is organized as follows. In Section 2 we provide further properties of the group G subject to Assumption 1 and the maximal operator $\mathcal{M}_G$. In the case of a translation-invariant measure μ, most of the results in this section are known and can be found in Chapter 2 of [10]. Section 3 is devoted to establishing Theorem 1. Finally, in Section 4, we provide basic information about $\mathcal{R}$-boundedness and prove Theorem 2.

2. Harmonic Analysis on Locally Compact Abelian Groups

We first provide some basic properties that follow directly from Assumption 1.

Proposition 1. *Suppose Assumption 1 is satisfied.*

(i) *For every* $x \in G$ *and* $k \in \mathbb{Z}$ *it holds* $\mu(x + U_k) > 0$*.*

(ii) *The interiors of the base sets* U_k *cover* G*, i.e.,* $\bigcup_{k\in\mathbb{Z}} \mathring{U}_k = G$*. In particular, for every compact* $K \subset G$ *there is* $k \in \mathbb{Z}$ *such that* $K \subset U_k$*.*

Proof. (i) By Assumption 1 (i) and the regularity of measure μ, we easily deduce $\mu(x + U_k) \to \mu(G)$ as $k \to \infty$. Since μ is nontrivial, we have $\mu(G) > 0$ and hence there exists $K \in \mathbb{Z}$ with $\mu(x + U_K) > 0$. Then for $k \in \mathbb{Z}$, Assumption 1 (iii) gives $k < \theta(k)$, which shows that for all $k \in \mathbb{Z}$ there exists $N \in \mathbb{N}$ with $\theta^N(k) \ge K$. Hence

$$0 < \mu(x + U_K) \le \mu(x + U_{\theta^N(k)}) \le A^N \mu(x + U_k),$$

proving the assertion.

(ii) It suffices to show that for every $k \in \mathbb{Z}$ we have $U_k \subset \mathring{U}_{\theta(k)}$ and then use property (i) of Assumption 1. So fix $k \in \mathbb{Z}$ and choose an open neighbourhood O of $0 \in G$ such that $O \subset U_k$. Then we have

$$U_k \subset O' := \bigcup_{x\in U_k} (x + O) \subset 2U_k.$$

Observe that O' is open, since it is the union of the open sets $x + O$. It follows $U_k \subset O' \subset U_{\theta(k)}$ and by definition of the interior even $U_k \subset O' \subset \mathring{U}_{\theta(k)}$, which is what we wanted to show.

For the assertion about the compact set K we note that $\{\mathring{U}_k\}_{k\in\mathbb{Z}}$ is an open cover of K and we thus find a finite subcover by compactness. But since the base sets U_k are nested, so are their interiors, and so the finite subcover consists only of the largest element. Hence there is $k \in \mathbb{Z}$ with $K \subset \mathring{U}_k \subset U_k$. □

One can define the uncentered maximal operator M_G in an analogous way, if one takes the supremum in (1.3) not only over all $k \in \mathbb{Z}$, but also over all centers $y \in G$ such that $x \in y + U_k$. By a similar reasoning as for the centered maximal operator, $M_G f$ is measurable. In fact, the uncentered maximal operator is comparable to the centered maximal operator.

Lemma 1. *Let $f \in L^1_{loc}(G)$. Then*

$$\mathcal{M}_G f \le M_G f \le A^2 \mathcal{M}_G f. \tag{2.1}$$

Moreover, for all $x \in G$ it holds

$$M_G^\theta f(x) := \sup_{k\in\mathbb{Z}} \sup_{y+U_k \ni x} \frac{1}{\mu(y+U_{\theta^2(k)})} \int_{y+U_k} |f|\, d\mu \le \mathcal{M}_G f(x).$$

Proof. The first inequality of (2.1) is obvious. For the second inequality, let $x, y \in G$ and $k \in \mathbb{Z}$ be such that $x \in y + U_k$. Hence, we obtain $x + U_k \subset y + U_{\theta(k)}$, and the doubling property yields

$$\mu(x + U_{\theta(k)}) \le A\mu(x + U_k) \le A\mu(y + U_{\theta(k)}) \le A^2 \mu(y + U_k).$$

On the other hand $y + U_k \subset x + U_{\theta(k)}$, and thus

$$\begin{aligned} \frac{1}{\mu(y+U_k)} \int_{y+U_k} |f|\, \mathrm{d}\mu &\le \frac{1}{\mu(y+U_k)} \int_{x+U_{\theta(k)}} |f|\, \mathrm{d}\mu \\ &\le \frac{A^2}{\mu(x+U_{\theta(k)})} \int_{x+U_{\theta(k)}} |f|\, \mathrm{d}\mu. \end{aligned}$$

Taking the supremum first on the right-hand side and then on the left-hand side yields (2.1). The second assertion follows analogously if one observes that $x + U_{\theta(k)} \subset y + U_{\theta^2(k)}$. □

As the measure μ possesses the doubling property, we expect the weak estimate

$$\mu(\{x \in G : \mathcal{M}_G f(x) > t\}) \le \frac{A}{t} \|f\|_1, \qquad t > 0, \tag{2.2}$$

and even the stronger form

$$\mu(\{x \in G : \mathcal{M}_G f(x) > t\}) \le \frac{2A}{t} \int_{\{|f|>t/2\}} |f|\, \mathrm{d}\mu, \qquad t > 0. \tag{2.3}$$

In order to show this, we need the following covering lemma due to Edwards and Gaudry [10] to apply the known technique from the $\mathbb{R}^n$-setting.

Lemma 2. *Let E be a subset of G and $k : E \to \mathbb{Z}$ a mapping bounded from above such that for every $k_0 \in \mathbb{Z}$ the set $\{x \in E : k(x) \ge k_0\}$ is relatively compact in G. Then there is a sequence $(x_n) \subset E$, finite or infinite, such that*

(i) *the sequence $(k_n) := (k(x_n))$ is decreasing,*
(ii) *the sets $x_n + U_{k_n}$ are pairwise disjoint and*
(iii) *$E \subset \bigcup (x_n + 2U_{k_n})$.*

Proof. The lemma has been proven in Lemma 2.2.1 of [10] in the case of an translation invariant measure μ. The proof in the more general case considered here needs some modifications.

If there is a finite sequence $x_1, \dots, x_m$ of points of E such that (ii) and (iii) are satisfied, then one can always achieve (i) by relabelling and there is nothing further to prove. Hence, assume that there is no such finite sequence.

Then, arguing exactly as in [10], we find a sequence of points $(x_n)_{n\in\mathbb{N}} \subset E$ such that (i) and (ii) are satisfied and such that $k_n = \max\{k(x) : x \in A_{n-1}\}$, where

$$A_n := E \setminus \left(\bigcup_{1\le l\le n} x_l + 2U_{k_l} \right).$$

It remains to proof (iii), *i.e*, that the intersection over all A_n is empty. Were this not the case, there would exist a point $x \in E$ belonging to every A_n, yielding $k_n \ge k(x)$ for all $n \in \mathbb{N}$. Therefore, by assumption, the set $M := \{x_n : n \in \mathbb{N}\}$ is relatively compact in G. Since $U_{k_n} \subset U_{k_1}$ and U_{k_1} is relatively compact, it follows that

$$F := \bigcup_{n\in\mathbb{N}} (x_n + U_{k_n}) \subset M + U_{k_1}$$

is relatively compact and so $\mu(F) \le \mu(\overline{F}) < \infty$. On the other hand, the compact set $\overline{M}$ is contained in a base set U_K, $K \in \mathbb{Z}$, by Proposition 1 (ii). Hence $x_n \in U_K$ for all $n \in \mathbb{N}$. Furthermore, by the monotonicity of θ, we find $N \in \mathbb{N}$ with $\theta^N(k(x)) \ge K$, and so $0 \in x_n + U_{\theta^N(k(x))}$. This shows

$$U_K \subset \big(x_n + U_{\theta^N(k(x))}\big) + U_K \subset x_n + 2U_{\theta^N(k(x))} \subset x_n + U_{\theta^{N+1}(k(x))}.$$

Since the $x_n + U_{k_n}$ are disjoint, this finally yields

$$\begin{aligned}\mu(F) &= \sum_{n\in\mathbb{N}} \mu(x_n + U_{k_n}) \ge \sum_{n\in\mathbb{N}} \mu(x_n + U_{k(x)}) \\ &\ge A^{-(N+1)} \sum_{n\in\mathbb{N}} \mu(x_n + U_{\theta^{N+1}(k(x))}) \ge A^{-(N+1)} \sum_{n\in\mathbb{N}} \mu(U_K) = \infty,\end{aligned}$$

since $\mu(U_K) > 0$ by Proposition 1 (i). This contradicts the finiteness of $\mu(F)$. Hence $\bigcap_{n\in\mathbb{N}} A_n$ is empty, finishing the proof. □

Theorem 3. *Let $q \in (1, \infty]$. Then the maximal operator $\mathcal{M}_G$ is bounded in $L^q(G)$. Furthermore, $\mathcal{M}_G$ is weakly bounded in $L^1(G)$,* i.e., *estimate* (2.2) *(and even* (2.3)*) holds true.*

Proof. Since $\mathcal{M}_G f$ is lower semi-continuous for $f \in L^1_{\mathrm{loc}}(G)$ and obviously $\mathcal{M}_G f(x) \le \|f\|_\infty$ almost everywhere, the maximal operator extends to a bounded operator in $L^\infty(G)$.

Let us now establish (2.2). Assume that $t > 0$ is such that $\mu(G) > \frac{A}{t}\|f\|_1$, since otherwise the assertion is trivial. As we want to apply Lemma 2, consider the set

$$E_t := \{x \in G : \mathcal{M}_G f(x) > t\}.$$

If E_t is empty, there is nothing to prove. Otherwise, choose a compact subset $E_t' \subset E_t$ and define a function $k : E_t' \to \mathbb{Z}$ via

$$k(x) := \max\left\{ k \in \mathbb{Z} : \frac{1}{\mu(x + U_k)} \int_{x+U_k} |f| \, \mathrm{d}\mu > t \right\}.$$

This mapping is certainly well-defined. Indeed, if there was no maximal $k \in \mathbb{Z}$, then we would find a sequence $(k_n) \subset \mathbb{Z}$ with $k_n \to \infty$ as $n \to \infty$ such that for all $n \in \mathbb{N}$ it holds

$$\frac{A}{t}\|f\|_1 \ge \frac{1}{t}\|f\|_1 \ge \frac{1}{t} \int_{x+U_{k_n}} |f| \, \mathrm{d}\mu \ge \mu(x + U_{k_n}) \to \mu(G), \qquad \text{as } n \to \infty,$$

contradicting our assumption.

We have to show that the mapping k is bounded from above. Assume again otherwise. Then we find $(x_n)_{n\in\mathbb{N}} \subset E_t'$ such that $k_n := k(x_n) \to \infty$ as $n \to \infty$. Since E_t' is compact, there is a $K \in \mathbb{Z}$ with $\bigcup_{n\in\mathbb{N}}\{x_n\} \subset E_t' \subset U_K$ by Proposition 1 ii. Taking sufficiently large $n \in \mathbb{N}$, we obtain $k_n \geq K$. Therefore $0 \in x_n + U_{k_n}$ and consequently $U_{k_n} \subset x_n + U_{\theta(k_n)}$. Hence, we see

$$\mu(U_{k_n}) \leq \mu(x_n + U_{\theta(k_n)}) \leq A\mu(x_n + U_{k_n}).$$

But then

$$\frac{A}{t}\|f\|_1 \geq \frac{A}{t}\int_{x_n+U_{k_n}} |f|\,\mathrm{d}\mu \geq A\mu(x_n+U_{k_n}) \geq \mu(U_{k_n}) \to \mu(G), \text{ as } n\to\infty,$$

yielding again a contradiction.

Since for every $k_0 \in \mathbb{Z}$ the set $\{x \in E_t' : k(x) \geq k_0\}$ is a subset of the compact E_t' and therefore relatively compact in G, we can invoke Lemma 2 to obtain a finite or infinite sequence of points x_n such that $E_t' \subset \bigcup (x_n + 2U_{k_n})$, but the sets $x_n + U_{k_n}$ are pairwise disjoint and it holds $\mu(x_n + U_{k_n}) < \frac{1}{t}\int_{x_n+U_{k_n}} |f|\,\mathrm{d}\mu$. Assume the obtained sequence to be infinite, the finite case being even easier. This yields

$$\begin{aligned}\mu(E_t') &\leq \sum_{n=1}^{\infty} \mu(x_n + 2U_{k_n}) \leq A\sum_{n=1}^{\infty}\mu(x_n+U_{k_n}) \\ &\leq \frac{A}{t}\sum_{n=1}^{\infty}\int_{x_n+U_{k_n}} |f|\,\mathrm{d}\mu \leq \frac{A}{t}\|f\|_1.\end{aligned}$$

Observe that this estimate is independent of the compact subset $E_t' \subset E_t$. Since E_t is open by the lower semi-continuity of the maximal operator and since the measure μ is inner regular, we may take the supremum over all compact subsets of E_t to obtain (2.2). Therefore $\mathcal{M}_G$ is continuous from $L^1(G)$ to $L^{1,\infty}(G)$.

Inequality (2.3) can be verified by a standard argument using (2.2), see e.g. Chapter I.3.1 of [25].

Since $\mathcal{M}_G$ is weakly bounded in $L^1(G)$ and bounded in $L^\infty(G)$, it is also bounded in $L^q(G)$ for $q \in (1,\infty)$ by the Marcinkiewicz interpolation theorem, see Appendix A in [10]. □

3. Muckenhoupt Weights

Assume that G is a locally compact abelian group with a measure μ satisfying Assumption 1.

Definition 1. Given a weight function $\omega \in L^1_{\mathrm{loc}}(G)$, we denote by μ_ω the measure defined via $\mu_\omega(E) := \int_E \omega\,\mathrm{d}\mu$ and by $L^q_\omega(G)$ the space of all measurable functions such that the q-norm with respect to the measure μ_ω is finite. Furthermore we denote by $\mathcal{M}_{G,\omega}$ the maximal operator defined as in (1.3) with respect to the measure μ_ω.

Definition 2. Let $q \in (1,\infty)$. A function $0 \leq \omega \in L^1_{loc}(G)$ is called an $A_q(G)$-*weight* if

$$\mathcal{A}_q(\omega) := \sup_{U\subset G}\left(\frac{1}{\mu(U)}\int_U \omega\,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}} < \infty, \tag{3.1}$$

where the supremum runs over all base sets $U \in G$. In that case, $\mathcal{A}_q(\omega)$ is called the $A_q(G)$-constant of ω. We say that ω belongs to the Muckenhoupt class $A_q(G)$ or even $\omega \in A_q(G)$ if it is an $A_q(G)$-weight.

Furthermore, we call a locally integrable, nonnegative function ω an $A_1(G)$-weight if there exists a constant $c \geq 0$ such that

$$\mathcal{M}_G\omega(x) \leq c\omega(x), \qquad \text{a.a. } x \in G. \tag{3.2}$$

The infimum over all these constants is called the $A_1(G)$-constant of ω and is denoted by $\mathcal{A}_1(\omega)$.

We call a constant $c = c(\omega) > 0$ that depends on $A_q(G)$-weights $A_q(G)$-*consistent*, if for each $d > 0$ we have

$$\sup\{c(\omega) : \omega \text{ is an } A_q(G)\text{-weight with } \mathcal{A}_q(\omega) < d\} < \infty.$$

Let us note some important observations on basic properties of the Muckenhoupt classes.

Proposition 2. (i) *Let $\omega \in A_q(G)$ for $q \in (1,\infty)$. Then the following hold true.*

(a) *$\omega \in A_p(G)$ for $p \in (q,\infty)$ and $\mathcal{A}_p(\omega)$ is $A_q(G)$-consistent. Here, even the end-point case $q = 1$ is allowed.*

(b) *$\omega^{-\frac{q'}{q}} \in A_{q'}(G)$, where q' is the Hölder conjugate of q. Moreover, $\mathcal{A}_{q'}(\omega^{-\frac{q'}{q}})$ is $A_q(G)$-consistent.*

(ii) *Let $0 \leq \omega \in L^1_{loc}(G)$ and let $q \in [1,\infty)$. Then $\omega \in A_q(G)$ if and only if there is an $A_q(G)$-consistent constant $c > 0$ such that for every nonnegative measurable function $f : G \to \mathbb{R}$ and every base set $U \subset G$ it holds*

$$\left(\frac{1}{\mu(U)}\int_U f\,d\mu\right)^q \leq \frac{c}{\mu_\omega(U)}\int_U f^q\omega\,d\mu. \tag{3.3}$$

(iii) (a) *Let $r, q \in [1,\infty)$ with $q < r$ and let $\omega_0 \in A_q(G)$, $\omega_1 \in A_1(G)$. Then it holds $\omega_0 \cdot \omega_1^{q-r} \in A_r(G)$ and*

$$\mathcal{A}_r(\omega_0 \cdot \omega_1^{q-r}) \leq \mathcal{A}_q(\omega_0)\mathcal{A}_1(\omega_1)^{r-q}.$$

(b) *Let $r, q \in (1,\infty)$ with $r < q$ and let $\omega_0 \in A_q(G)$, $\omega_1 \in A_1(G)$. Then it holds $(\omega_0^{r-1} \cdot \omega_1^{q-r})^{1/(q-1)} \in A_r(G)$ and*

$$\mathcal{A}_r((\omega_0^{r-1} \cdot \omega_1^{q-r})^{1/(q-1)}) \leq \mathcal{A}_q(\omega_0)^{\frac{r-1}{q-1}}\mathcal{A}_1(\omega_1)^{\frac{q-r}{q-1}}.$$

(iv) *Let $q \in [1,\infty)$ and $\omega \in A_q(G)$. Then*

(a) *the measure μ_ω is regular and has the doubling property, i.e.,*

$$\mu_\omega(x + U_{\theta(k)}) \leq c_\omega\mu_\omega(x + U_k)$$

for all $x \in G$ and $k \in \mathbb{Z}$, where $c_\omega > 0$ is an $A_q(G)$-consistent constant,

(b) *slightly more general, for any base set U and any measurable subset $S \subset U$ we have*

$$\left(\frac{\mu(S)}{\mu(U)}\right)^q \leq c\frac{\mu_\omega(S)}{\mu_\omega(U)}, \tag{3.4}$$

where $c > 0$ is the bound appearing in (3.3),

(c) *it holds $L^\infty(G) = L^\infty_\omega(G)$ with equal norms,*

(d) *$\mathcal{M}_{G,\omega}$ is bounded in $L^p_\omega(G)$ for all $1 < p \leq \infty$ and weakly bounded in $L^1_\omega(G)$ with an $A_q(G)$-consistent bound.*

Proof. Parts (i) and (ii) are analogous to [13]. The respective consistencies are apparent from the proof given there. Part (iii) is analogous to Lemma 2.1 of [9]. Thus, we concentrate on part (iv).

Regularity follows by Lebesgue's Theorem on Dominated Convergence. To verify the doubling property, simply use (3.3) with $U = x + U_{\theta(k)}$ and $f = \chi_{x+U_k}$. Since μ has the doubling property with doubling constant A, we obtain (iva) with $c_\omega = cA^q$.

For (ivb), we argue analogously, using (3.3) with $f = \chi_S$.

To show (ivc), recall that the norm on $L^\infty_\omega(G)$ can be represented via

$$\|f\|_{L^\infty_\omega(G)} = \sup\{r \in \mathbb{R} : \mu_\omega(\{x \in G : f(x) > r\}) > 0\},$$

and a similar expression for the norm on $L^\infty(G)$, if we replace the measure μ_ω by the measure μ. Since μ_ω is absolutely continuous with respect to μ, clearly $\|f\|_{L^\infty_\omega(G)} \le \|f\|_{L^\infty(G)}$. Moreover, $\omega > 0$ almost everywhere, excepting the trivial case $\omega = 0$. Indeed, if $\omega = 0$ on a set S such that $\mu(S) > 0$, we get in virtue of (3.4) that $\mu_\omega(U) = 0$ for every base set U containing S. If S is not contained in any base set, then consider the set $\tilde{S} := S \cap U$ for some base set U large enough such that $\mu(\tilde{S}) > 0$, which certainly exists, since otherwise

$$\mu(S) = \mu\left(\bigcup_{k\in\mathbb{Z}}(S \cap U_k)\right) \le \sum_{k\in\mathbb{Z}} \mu(S \cap U_k) = 0.$$

Hence, $\omega = 0$ almost everywhere on every base set containing $\tilde{S}$ and thus on the whole group G. This shows that for every nontrivial Muckenhoupt weight ω we have $\omega > 0$ almost everywhere. Thus, μ is absolutely continuous with respect to μ_ω. Consequently $\|f\|_{L^\infty(G)} = \|f\|_{L^\infty_\omega(G)}$.

The boundedness of the maximal operators follows by Theorem 3. Marcinkiewicz' interpolation theorem yields together with part (iva) the $A_q(G)$-consistency of the bound. $\square$

The Muckenhoupt weights can be characterized as those weight functions such that the maximal operator is weakly bounded in the weighted function space $L^q_\omega(G)$. In fact, Theorem 1 states that for $q \in (1, \infty)$ the maximal operator is bounded in $L^q_\omega(G)$ even in the strong sense. However, we first focus on the weak boundedness, which is true also for $q = 1$.

Theorem 4. *Let $0 \le \omega \in L^1_{loc}(G)$ and let $q \in [1, \infty)$. Then $\omega \in A_q(G)$ if and only if $\mathcal{M}_G$ is bounded from $L^q_\omega(G)$ to $L^{q,\infty}_\omega(G)$ with an $A_q(G)$-consistent bound.*

Proof. Assume $\omega \in A_q(G)$. We can apply Proposition 2 (iv) to obtain that $\mathcal{M}_{G,\omega}$ is weakly bounded in $L^1_\omega(G)$ with an $A_q(G)$-consistent bound. Therefore, the "only if" implication follows by the arguments given in Chapter V.2.2 of [25].

Conversely, assume that $\mathcal{M}_G$ is bounded from L^q_ω to $L^{q,\infty}_\omega$. Let $f \ge 0$ be measurable and let $U \subset G$ be a base set. If

$$(f_U) := \frac{1}{\mu(U)} \int_U f \, \mathrm{d}\mu = 0,$$

there is nothing left to prove. Hence, assume $(f_U) > 0$ and observe that for every $x \in U$ it holds $(f_U) \le M_G f(x) \le A^2 \mathcal{M}_G f(x)$. Fixing $0 < t < (f_U)$, we obtain

$$\begin{aligned} U &= \{x \in U : \mathcal{M}_G f(x) \ge (f_U)/A^2\} \\ &\subset \{x \in U : \mathcal{M}_G f(x) > t/A^2\} \subset \{x \in G : \mathcal{M}_G f(x) > t/A^2\}, \end{aligned}$$

and by the weak boundedness of the maximal operator we obtain

$$\mu_\omega(U) \le \frac{cA^{2q}}{t^q} \int_U |f|^q \,\mathrm{d}\mu_\omega.$$

Letting $t \to (f_U)$, we finally see

$$(f_U)^q \mu_\omega(U) \le cA^{2q} \int_U |f|^q \,\mathrm{d}\mu_\omega,$$

and in virtue of Proposition 2 (ii) we obtain $\omega \in A_q(G)$. □

If we want to strengthen Theorem 4 towards strong boundedness, we will necessarily have to exclude the case $q = 1$: There are counterexamples even for the group $G = \mathbb{R}^n$. Take for example $\omega = 1$. It is easy to see that applying the maximal operator to a nontrivial nonnegative integrable function never yields an integrable function.

However, if $1 < q = p$, then we do obtain such a strong estimate. In the classical setting $G = \mathbb{R}^n$, this is called the Muckenhoupt theorem. It is usually proven via the so-called reverse Hölder inequality (cf. [14], [25]), which in turn shows for $q \in (1, \infty)$ that $\omega \in A_q(G)$ implies $\omega \in A_p(G)$ for some smaller $p < q$. Then the Marcinkiewicz interpolation theorem may be applied to show the assertion. Unfortunately, the proof of the reverse Hölder inequality heavily relies on the existence of dyadic cubes. In our situation, we lack of such a concept. However, Jawerth [16] found a different approach avoiding the reverse Hölder inequality and hence suitable to adapt to our situation. Later, Lerner [19] significantly simplified the argument.

Proof of Theorem 1. Let $f \in L^q_\omega(G)$ and assume that $U := x + U_k$, $x \in G$, $k \in \mathbb{Z}$ is a base set. Define

$$\mathcal{A}_{q,U}(\omega) := \left(\frac{1}{\mu(U)} \int_U \omega \,\mathrm{d}\mu\right) \left(\frac{1}{\mu(U)} \int_{x+U_{\theta^2(k)}} \omega^{-\frac{q'}{q}} \,\mathrm{d}\mu\right)^{\frac{q}{q'}}.$$

Note that $\mathcal{A}_{q,U}(\omega)$ can be estimated by $A^{2q}\mathcal{A}_q(\omega)$. We write $\nu := \omega^{-q'/q}$ and observe that $\nu \in A_{q'}(G)$ by Proposition 2 (ii). We calculate

$$\begin{aligned}
\frac{1}{\mu(U)} \int_U |f| \,\mathrm{d}\mu &= \mathcal{A}_{q,U}(\omega)^{\frac{q'}{q}} \left(\frac{\mu(U)}{\mu_\omega(U)} \left(\frac{1}{\mu_\nu(x + U_{\theta^2(k)})} \int_U |f|\nu^{-1} \,\mathrm{d}\mu_\nu\right)^{\frac{q}{q'}}\right)^{\frac{q'}{q}} \\
&\le A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\frac{\mu(U)}{\mu_\omega(U)} \left(\inf_{y\in U} M^\theta_{G,\nu}(f\nu^{-1})(y)\right)^{\frac{q}{q'}}\right)^{\frac{q'}{q}} \\
&\le A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\frac{1}{\mu_\omega(U)} \int_U M^\theta_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}} \,\mathrm{d}\mu\right)^{\frac{q'}{q}} \\
&\le A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\frac{1}{\mu_\omega(U)} \int_U \mathcal{M}_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}} \,\mathrm{d}\mu\right)^{\frac{q'}{q}},
\end{aligned}$$

where the last estimate has been proven in Lemma 1. Therefore we deduce

$$\mathcal{M}_G f(x) \le A^{2q'} \mathcal{A}_q(\omega)^{\frac{q'}{q}} \left(\mathcal{M}_{G,\omega}(\mathcal{M}_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}} \omega^{-1})(x)\right)^{\frac{q'}{q}}.$$

Consequently,

$$\begin{aligned}\|\mathcal{M}_G f\|_{L^q_\omega(G)} &\le A^{2q'}\mathcal{A}_q(\omega)^{\frac{q'}{q}}\|\mathcal{M}_{G,\omega}(\mathcal{M}_{G,\nu}(f\nu^{-1})^{\frac{q}{q'}}\omega^{-1})\|^{\frac{q'}{q}}_{L^{q'}_\omega(G)}\\ &\le A^{2q'}\mathcal{A}_q(\omega)^{\frac{q'}{q}}\|\mathcal{M}_{G,\omega}\|^{\frac{q'}{q}}_{L^{q'}_\omega(G)}\|\mathcal{M}_{G,\nu}(f\nu^{-1})\|_{L^q_\nu(G)}\\ &\le A^{2q'}\mathcal{A}_q(\omega)^{\frac{q'}{q}}\|\mathcal{M}_{G,\omega}\|^{\frac{q'}{q}}_{L^{q'}_\omega(G)}\|\mathcal{M}_{G,\nu}\|_{L^q_\nu(G)}\|f\|_{L^q_\omega(G)}.\end{aligned} \tag{3.5}$$

Here, $\|\mathcal{M}_{G,\omega}\|_{L^{q'}_\omega(G)}$ is the operator norm which is bounded by an $A_q(G)$-consistent bound $C(\omega)$ by Proposition 2 (iv). Similarly, $\|\mathcal{M}_{G,\nu}\|_{L^q_\nu(G)}$ is bounded by an $A_q(G)$-consistent bound $\tilde{C}(\omega)$ by Proposition 2 (ii) and (iv). Hence, the assertion follows. □

Remark 3. As remarked in [19], in the classical case $G = \mathbb{R}^n$ the bounds of $\mathcal{M}_{\mathbb{R}^n,\omega}$ and $\mathcal{M}_{\mathbb{R}^n,\nu}$ are uniform in $\omega \in A_q(\mathbb{R}^n)$ by the Besicovič covering theorem. Therefore, the bound of the maximal operator in (3.5) reduces to $cA_q(\omega)^{q'/q}$ with $c = c(n,q) > 0$ in this case. Buckley [3] showed that this estimate is sharp, *i.e.*, the exponent q'/q is the best possible. It would be interesting to investigate if one can achieve a sharp result also in the more general setup considered here.

4. $\mathcal{R}$-boundedness and Extrapolation Theorem

This section is devoted to establishing an extrapolation theorem generalizing the classical extrapolation theorem due to García-Cuerva and Rubio de Francia towards $\mathcal{R}$-boundedness. Therefore, we first provide some basic facts about $\mathcal{R}$-bounded families of operators.

Definition 3. We call the sequence of functions $(r_j)_{j\in\mathbb{N}}$ defined via

$$\begin{aligned} r_j &: [0,1] \to \{-1,1\},\\ r_j(t) &:= \operatorname{sgn}[\sin(2^{j-1}\pi t)],\end{aligned}$$

the sequence of *Rademacher functions.*

Remark 4. Note that the Rademacher functions are symmetric, independent and $\{-1,1\}$-valued random variables on the probability space $([0,1],\mathcal{B},\lambda)$, where $\mathcal{B}$ is the Borel algebra on $[0,1]$ and λ is the corresponding Borel measure. In fact, all arguments used in this section can be be transferred from Rademacher functions to symmetric, independent, $\{-1,1\}$-valued random variables on $[0,1]$ without any changes.

Definition 4. Let $\mathcal{X}$ be a Banach space. A subset $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ is called $\mathcal{R}$-bounded, if there exists a constant $c > 0$ such that

$$\int_0^1 \Big\|\sum_{j=1}^n r_j(t)T_jx_j\Big\|_{\mathcal{X}} \,\mathrm{d}t \le c\int_0^1 \Big\|\sum_{j=1}^n r_j(t)x_j\Big\|_{\mathcal{X}} \,\mathrm{d}t \tag{4.1}$$

for all $T_1,\dots,T_N \in \mathcal{T}$, $x_1,\dots,x_n \in \mathcal{X}$ and $n \in \mathbb{N}$. Here, $(r_j)_{j\in\mathbb{N}}$ is the sequence of Rademacher functions.

The smallest constant $c > 0$ such that (4.1) holds is called *$\mathcal{R}$-bound of $\mathcal{T}$* and is denoted by $\mathcal{R}_1(\mathcal{T})$.

For $1 \leq p < \infty$, we can replace the condition (4.1) in Definition 4 by

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) T_j x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t \leq \mathcal{R}_p(\mathcal{T}) \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t, \tag{4.2}$$

due to the following lemma, which is known as *Kahane's inequality.*

Lemma 3. *Let $(r_j)_{j\in\mathbb{N}}$ be the sequence of Rademacher functions. Then there is a constant $k_p > 0$ such that for every Banach space $\mathcal{X}$ and for all $x_1, \dots x_n \in \mathcal{X}$*

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \mathrm{d}t \leq \Big(\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}}^p \mathrm{d}t \Big)^{\frac{1}{p}} \leq k_p \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \mathrm{d}t.$$

Hence, (4.1) *holds with a bound $\mathcal{R}_1(\mathcal{T}) := k_p \mathcal{R}_p(\mathcal{T})^{\frac{1}{p}}$ if* (4.2) *holds with a bound $\mathcal{R}_p(\mathcal{T})$, and* (4.2) *holds with a bound $\mathcal{R}_p(\mathcal{T}) := (k_p \mathcal{R}_1(\mathcal{T}))^p$ if* (4.1) *holds with a bound $\mathcal{R}_1(\mathcal{T})$.*

Proof. See Theorem 11.1 in [7]. □

In the particular case that $\mathcal{X}$ is an $L^q(X, \mu_X)$-space, where (X, μ_X) is a measure space, we can give a characterization of $\mathcal{R}$-boundedness that is much easier to handle. It relies on the following *Khinchin's inequality.*

Lemma 4. *Let $0 < q < \infty$ and $(r_j)_{j\in\mathbb{N}}$ be the sequence of Rademacher functions. Then there is a constant $c_q > 0$ such that*

$$c_q^{-1} \Big(\sum_{j=1}^n |a_j|^2 \Big)^{\frac{1}{2}} \leq \Big(\int_0^1 \Big| \sum_{j=1}^n r_j(t) a_j \Big|^q dt \Big)^{\frac{1}{q}} \leq c_q \Big(\sum_{j=1}^n |a_j|^2 \Big)^{\frac{1}{2}}, \tag{4.3}$$

for all $a_1, \dots, a_n \in \mathbb{C}$ and all $n \in \mathbb{N}$.

Proof. See Theorem 1.10 in [7]. □

Proposition 3. *Let $(X, \mathcal{A}, \mu_X)$ be a measure space, $q \in (1, \infty)$ and write $\mathcal{X} := L^q(X, \mu_X)$. Then $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ is $\mathcal{R}$-bounded if and only if there is a constant $c > 0$ such that*

$$\Big\| \Big(\sum_{j=1}^n |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{X}} \leq c \cdot \Big\| \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{X}}, \tag{4.4}$$

for all $T_1, \dots, T_n \in \mathcal{T}$, $f_1, \dots, f_n \in \mathcal{X}$ and $n \in \mathbb{N}$.

Proof. See e.g. Lemma 4.2 of [12]. □

Remark 5. If in the situation of Proposition 3 the constant c appearing in (4.4) is $A_q(G)$-consistent, then also the $\mathcal{R}$-bound of $\mathcal{T}$ is $A_q(G)$-consistent. Indeed, from the proof of Lemma 4.2 in [12] it is apparent that $\mathcal{R}_q(\mathcal{T}) = c_q^2 c$ is $A_q(G)$-consistent; here, c_q is the constant from Khinchin's inequality (4.3) which is independent of ω. But since $\mathcal{R}_1(\mathcal{T}) = k_q \mathcal{R}_q(\mathcal{T})^{1/q}$ by Lemma 3, and since k_q is independent of the underlying Banach space and therefore in particular $A_q(G)$-consistent, we see that $\mathcal{R}_1(\mathcal{T})$ is $A_q(G)$-consistent.

Proposition 3 suggests that a vector-valued extrapolation theorem is sufficient to pass from uniform to $\mathcal{R}$-bounds. In fact, in the classical situation $G = \mathbb{R}^n$, such a vector-valued theorem has been proven by García-Cuerva and Rubio de Francia already in their book [13]. Since then, their original arguments have been improved and simplified several times, see for example [5] and [9]. We will state here a more abstract version following the latter approach.

Proposition 4. *Let G be a locally compact abelian group satisfying Assumption 1 and let $\Omega \subset G$ be measurable. Moreover, let $r \in [1,\infty)$ and assume that there is*

$$\mathcal{F} \subset \{(f,g) : f,g : \Omega \to \mathbb{R} \textit{ are nonnegative, measurable functions}\},$$

such that for every $\upsilon \in A_r(G)$,

$$\|g\|_{L^r_\upsilon(\Omega)} \leq \tilde{c}\|f\|_{L^r_\upsilon(\Omega)}, \qquad (f,g) \in \mathcal{F}, \tag{4.5}$$

with an $A_r(G)$-consistent constant $\tilde{c} = \tilde{c}(\upsilon) > 0$. Then for every $q \in (1,\infty)$ and every $\omega \in A_q(G)$,

$$\|g\|_{L^q_\omega(\Omega)} \leq c\|f\|_{L^q_\omega(\Omega)}, \qquad (f,g) \in \mathcal{F}, \tag{4.6}$$

with an $A_q(G)$-consistent constant $c = c(q,\omega) > 0$.

If the constant $\tilde{c}(\omega)$ appearing in (4.5) is of the form $N(A_r(\omega))$, where N is an increasing function, then one obtains for the constant in (4.6)

$$c(q,\omega) = \begin{cases} N(A_q(\omega)(2\|\mathcal{M}_G\|_{L^q_\omega(G)})^{r-q}), & \text{if } q < r, \\ N(A_q(\omega)^{\frac{r-1}{q-1}}(2\|\mathcal{M}_G\|_{L^{q'}_\nu(G)})^{\frac{q-r}{q-1}}), & \text{if } q > r. \end{cases} \tag{4.7}$$

where $\nu := \omega^{-q'/q}$.

Proof. See Theorem 3.1 of [9] for a proof in the classical case $\mathbb{R}^n$. A quick inspection shows that the arguments given there only use elementary calculations, Hölder's inequality, the theorem of Hahn-Banach, the factorization properties stated in Proposition 2 (iii) and, as the main ingredient, the boundedness of the maximal operator in L^q_ω and $L^{q'}_\nu$. Hence, the assertion carries over to our setting. □

Remark 6. The bound obtained in (4.7) is of particular interest in regard of Remark 3, since it provides sharp bounds for the extrapolation theorem in the classical case $G = \mathbb{R}^n$ due to the sharp dependence of the maximal operator on the Muckenhoupt weight, as pointed out by Dragičević, Grafakos, Pereyra and Petermichl [8].

Remark 7. Proposition 4 contains an extrapolation theorem on locally compact abelian groups in the style of García-Cuerva and Rubio de Francia.

(i) Choose

$$\mathcal{F}_{cl} := \{(|f|,|Tf|) : f : \Omega \to \mathbb{R} \text{ continuous with compact support}\}.$$

If $T : L^r_\upsilon(\Omega) \to L^r_\upsilon(\Omega)$ is bounded with an $A_r(G)$-consistent bound, then we always have

$$\|g\|_{L^r_\upsilon(\Omega)} = \|Tf\|_{L^r_\upsilon(\Omega)} \leq c\|f\|_{L^r_\upsilon(\Omega)}, \qquad (f,g) \in \mathcal{F}_{cl},$$

and thus Proposition 4 gives us

$$\|Tf\|_{L^q_\omega(\Omega)} = \|g\|_{L^r_\omega(\Omega)} \leq c\|f\|_{L^q_\omega(\Omega)}, \qquad (f,g) \in \mathcal{F}_{cl},$$

with an $A_q(G)$-consistent constant $c = c(q,\omega) > 0$. By density, this yields the $A_q(G)$-consistent boundedness of T in $L^q_\omega(\Omega)$.

(ii) We also get a vector-valued version of Proposition 4, *i.e.*, under the assumption of the theorem we have for all $p, q \in (1,\infty)$ and for all $\omega \in A_q(G)$

$$\left\| \left(\sum_{j=1}^{n} g_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)} \le c \left\| \left(\sum_{j=1}^{n} f_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)},$$

for all finite sequences $\{(f_j, g_j)\}_{j=1}^n \subset \mathcal{F}$, where $c = c(q,p,\omega) > 0$ is $A_q(G)$-consistent. To see this, consider

$$\mathcal{F}_p := \left\{ (F,G) = \left(\left(\sum_{j=1}^{n} f_j^p \right)^{1/p}, \left(\sum_{j=1}^{n} g_j^p \right)^{1/p} \right) : \{(f_j, g_j)\}_{j=1}^n \subset \mathcal{F} \right\},$$

and observe that Proposition 4 applied with q replaced by p gives for all $\nu \in A_p(G)$ and $(F,G) \in \mathcal{F}_p$

$$\|G\|^p_{L^p_\nu(\Omega)} = \sum_{j=1}^{n} \int_\Omega g_j^p \, \mathrm{d}\mu_\nu \le c \sum_{j=1}^{n} \int_\Omega f_j^p \, \mathrm{d}\mu_\nu \le c \|F\|^p_{L^p_\nu(\Omega)},$$

with an $A_p(G)$-consistent constant $c = c(p,\nu) > 0$. Thus, taking the pth-root, we obtain $\|G\|_{L^p_\nu(G)} \le c\|F\|_{L^p_\nu(G)}$ for all $(F,G) \in \mathcal{F}_p$. If we apply now Proposition 4 again, but this time with exponents $r = p$, $q = q$ and $\mathcal{F} = \mathcal{F}_p$, we obtain

$$\left\| \left(\sum_{j=1}^{n} g_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)} = \|G\|_{L^q_\omega(\Omega)} \le c\|F\|_{L^q_\omega(\Omega)} = c \left\| \left(\sum_{j=1}^{n} f_j^p \right)^{1/p} \right\|_{L^q_\omega(\Omega)},$$

with an $A_q(G)$-consistent constant $c = c(q,p,\omega) > 0$.

We can finally give the proof of our main theorem.

Proof of Theorem 2. We will choose

$$\mathcal{F} := \{(|f|, |Tf|) : f : \Omega \to \mathbb{R} \text{ continuous with compact support}, \ T \in \mathcal{T}\}.$$

Then using the vector-valued extrapolation estimate in Remark 7 (ii) with $p = 2$, we obtain

$$\left\| \left(\sum_{j=1}^{n} |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q_\omega(\Omega)} \le c \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q_\omega(\Omega)},$$

for all $T_1, \ldots T_n \in \mathcal{T}$, $f_1, \ldots, f_n$ and all $n \in \mathbb{N}$. Hence, Proposition 3 yields the $\mathcal{R}$-boundedness of $\mathcal{T}$ and Remark 5 shows that the $\mathcal{R}$-bound is $A_q(G)$-consistent. $\square$

References

[1] E. M. Alfsen. A Simplified Constructive Proof of the Existence and Uniqueness of Haar measure. *Math. Scand.*, 12:106–116, 1963.

[2] E. Berkson and T. A. Gillespie. On Restrictions of Multipliers in Weighted Settings. *Indiana Univ. Math. J.*, 52(4):927–961, 2003.

[3] S. M. Buckley. Estimates for Operator Norms on Weighted Spaces and Reverse Jensen Inequalities. *Trans. Amer. Math. Soc.*, 340(1):253–272, 1993.

[4] H. Cartan. Sur la mesure de Haar. *C. R. Acad. Sci. Paris*, 211:759–762, 1940.

[5] D. Cruz-Uribe, J. M. Martell, and C. Pérez. Extensions of Rubio de Francia's Extrapolation Theorem. *Collect. Math.*, (Vol. Extra):195–231, 2006.

[6] R. Denk, M. Hieber, and J. Prüß. $\mathcal{R}$-Boundedness, Fourier Multipliers and Problems of Elliptic and Parabolic Type. *Mem. Amer. Math. Soc.*, 166(788), 2003.

[7] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely Summing Operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[8] O. Dragičević, L. Grafakos, M. C. Pereyra, and S. Petermichl. Extrapolation and Sharp Norm Estimates for Classical Operators on Weighted Lebesgue Spaces. *Publ. Mat.*, 49(1):73–91, 2005.

[9] J. Duoandikoetxea. Extrapolation of Weights Revisited: New Proofs and Sharp Bounds. *J. Funct. Anal.*, 260(6):1886–1901, 2011.

[10] R. E. Edwards and G. I. Gaudry. *Littlewood-Paley and Multiplier Theory*. Springer-Verlag, Berlin, 1977. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 90.

[11] R. Farwig and H. Sohr. Weighted L^q-Theory for the Stokes Resolvent in Exterior Domains. *J. Math. Soc. Japan*, 49(2):251–288, 1997.

[12] A. Fröhlich. The Stokes Operator in Weighted L^q-Spaces. II. Weighted Resolvent Estimates and Maximal L^p-Regularity. *Math. Ann.*, 339(2):287–316, 2007.

[13] J. García-Cuerva and J. L. Rubio de Francia. *Weighted Norm Inequalities and Related Topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985.

[14] L. Grafakos. *Modern Fourier Analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009.

[15] A. Haar. Der Maßbegriff in der Theorie der kontinuierlichen Gruppen. *Ann. of Math. (2)*, 34(1):147–169, 1933.

[16] B. Jawerth. Weighted Inequalities for Maximal Operators: Linearization, Localization and Factorization. *Amer. J. Math.*, 108(2):361–414, 1986.

[17] P. C. Kunstmann and L. Weis. Maximal L_p-Regularity for Parabolic Equations, Fourier Multiplier Theorems and H^∞-Functional Calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.

[18] M. Kyed. Maximal Regularity of the Time-Periodic Linearized Navier-Stokes System. *J. Math. Fluid Mech.*, 16(3):523–538, 2014.

[19] A. K. Lerner. An Elementary Approach to Several Results on the Hardy-Littlewood Maximal Operator. *Proc. Amer. Math. Soc.*, 136(8):2829–2833, 2008.

[20] M. Milman. A Note on Gehring's Lemma. *Ann. Acad. Sci. Fenn. Math.*, 21(2):389–398, 1996.

[21] B. Muckenhoupt. Weighted Norm Inequalities for the Hardy Maximal Function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[22] W. Rudin. *Fourier Analysis on Groups*. Interscience Tracts in Pure and Applied Mathematics, No. 12. Interscience Publishers, New York-London, 1962.

[23] J. Sauer. Maximal Regularity of the Spatially Periodic Stokes Operator and Application to Nematic Liquid Crystal Flows. *Czechoslovak Math. J.*, 66 (1):41–55, 2016.

[24] J. Sauer. Weighted Resolvent Estimates for the Spatially Periodic Stokes Equations. *Ann. Univ. Ferrara*, 61(2):333–354, 2015.

[25] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[26] E. M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[27] A. Weil. *L'intégration dans les groupes topologiques et ses applications*. Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940.

Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, D-64283 Darmstadt,

E-mail address: jsauer@mathematik.tu-darmstadt.de

JOURNAL OF ELLIPTIC AND PARABOLIC EQUATIONS

VOLUME 1, p. 231-414

CONTENTS

JOURNAL OF ELLIPTIC AND PARABOLIC EQUATIONS

VOLUME 1, p. 231-414

CONTENTS

ORTHOGONAL EDITIONS

For subscriptions, instructions to authors see :

http://www.orthogonal-editions.com/